Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 18, 2004*
Accepted: *July 23, 2004*

Hidden exceptional global symmetries in 4d CFTs

Sebastián Franco[a], Amihay Hanany[ab] and Pavlos Kazakopoulos[a*]

[a] *Center for Theoretical Physics, Massachusetts Institute of Technology Cambridge, MA 02139, U.S.A.*
[b] *Institute for Advanced Study Princeton NJ 08540 U.S.A.*
E-mail: sfranco@mit.edu, sfrancohanany@mit.edu, sfranconoablake@mit.edu

Abstract: We study four dimensional $\mathcal{N} = 1$ gauge theories that arise on the worldvolume of D3-branes probing complex cones over del Pezzo surfaces. Global symmetries of the gauge theories are made explicit by using a correspondence between bifundamental fields in the quivers and divisors in the underlying geometry. These global symmetries are hidden, being unbroken when all inverse gauge couplings of the quiver theory vanish. In the broken phase, for finite gauge couplings, only the Cartan subalgebra is manifest as a global symmetry. Superpotentials for these models are constructed using global symmetry invariants as their building blocks. Higgsings connecting theories for different del Pezzos are immediately identified by performing the appropriate higgsing of the global symmetry groups. The symmetric properties of the quivers are also exploited to count the first few dibaryon operators in the gauge theories, matching their enumeration in the AdS duals.

Keywords: D-branes, Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence.

*Research supported in part by the CTP and the LNS of MIT. Further support is granted from the U.S. Department of Energy under cooperative agreements #DE-FC02-94ER40818 and #DE-FG02-95ER40893, and by BSF – American-Israeli Bi-National Science Foundation. A.H. is also supported by the Reed Fund Award and a DOE OJI award.

Contents

---

1. Introduction

One of the possible ways in which gauge theories can be engineered in String Theory is by considering the low energy limit of D-branes probes on singularities. When D3-branes are used to probe local Calabi-Yau threefolds, the resulting theory on their worldvolume is a $\mathcal{N}=1$, $d=4$ quiver gauge theory.

A large class of interesting geometries is given by toric singularities. The gauge group, matter content and interaction superpotential of the gauge theory are dictated by the underlying geometry. A systematic procedure for obtaining the gauge theory consists in realizing the singularity as a partial resolution of an abelian orbifold [1, 2, 3]. A remarkable fact is that the gauge theory living on the D3-brane probes is not unique. This phenomenon is a manifestation of a full IR equivalence between different field theories and is called Toric Duality [2, 3]. It has been shown [4, 5] that toric dual theories are non-trivial realizations of the well known Seiberg duality [6].

A particular class of interesting singularities is the one of complex cones over del Pezzo surfaces. Recently, a map between bifundamental fields in certain del Pezzo quivers and 2-cycles in the geometry has been established [7]. Furthermore, a set of n conserved global U(1) currents, together with corresponding charges of the bifundamental fields under these currents were identified. The existence of these currents is encouraging and actually fits the expectation for an enhanced global E_n symmetry by identifying them with the Cartan elements of the E_n symmetry. The main objective of this paper is to exploit the map between geometric properties and matter fields in order to make the symmetries of the underlying geometry manifest at the level of the quiver gauge theory on the D-branes. This is not an obvious task since, as we will see in coming sections, a given irreducible representation of these global E_n symmetry groups may be formed by fields charged under different gauge groups. What this implies is that quiver gauge quantum numbers actually break the global E_n symmetry. The relevant deformations associated to each of these quiver gauge groups is their corresponding gauge couplings. As a result, we would expect that when all inverse gauge couplings vanish and the gauge group quantum numbers are absent, the E_n symmetry will be restored. This leads to the natural identification that the gauge couplings are in fact Cartan elements of an adjoint valued complex scalar field of E_n. The gauge couplings are Kähler moduli and we expect an E_n enhanced symmetry points at the origin of the Kähler moduli space.

The structure that is found here, i.e. the grouping of matter fields into E_n representations and the resulting successful reformulation of several properties of the gauge theories in the language of group theory of these exceptional Lie algebras seems to point towards the existence of a fixed point with enhanced exceptional global symmetry for each of the del Pezzo theories. The enhanced E_n symmetry is hidden in the sense that it does not appear in the perturbative lagrangian definition of the theories, and one can only argue for the existence of a superconformal fixed point where the symmetry is realized. For $n = 6, 7, 8$ this is of course to be expected, since there are no known lagrangians that manifest this type of global symmetry. Examples of hidden global symmetry enhancement have been discovered in three and five dimensional gauge theories. In three dimensions, a $N = 4$ U(1) gauge theory with 2 charged hypermultiplets flows to a fixed point with enhanced SU(2) global symmetry and infinite gauge coupling [8]. More generally, the theories on D2 branes probing ADE singularities of an ALE space are argued to possess a fixed point (at infinite gauge coupling) with hidden global symmetry of the corresponding ADE type [9]. In the T-dual picture one has three-branes suspended between NS fivebranes and the global symmetry can be seen as the gauge symmetry living on the NS fivebranes [10]. In five dimensions, the $\mathcal{N} = 1$ gauge theory on a D4 brane probing a certain type-I$'$ background is shown to have a fixed point with enhanced E_n global symmetry, depending on the number of D8 branes in the background [11]. Here also the fixed point resides at infinite gauge coupling. More examples can be found in [12] (see also [13]). The study of hidden symmetry enhancement in five dimensions can also be approached through (p, q) web techniques as in [14, 15] and the symmetry is made manifest in the string theory construction with the introduction of 7-branes [16]. The theories we have at hand bear striking similarities to these examples, namely the appearance of the E_n Lie groups as hidden global symmetries and the realization of the enhanced symmetry only at zero inverse gauge coupling. On the other hand, all the aforementioned examples are theories with eight supercharges in contrast to the four supercharges of the theories on D3-branes probing cones over del Pezzo surfaces. To the best of our knowledge, this is the first example of gauge theories with $\mathcal{N} = 1$ supersymmetry in four dimensions that exhibit a hidden exceptional global symmetry.

There are several problems that can be addressed once quivers are classified using global symmetries. A specially challenging task when deriving gauge theories that live on D-branes on singularities is the determination of the corresponding superpotentials. After organizing the matter content into representations, while ignoring their quiver gauge quantum numbers, the building blocks for superpotentials are given by invariant combinations of such irreducible representations. This is a key observation since, with this amount of supersymmetry, superpotentials are only affected by closed string complex moduli and will not be affected by changing Kähler moduli. As a result it is possible to compute the superpotentials at the enhanced point were the full E_n symmetry is enhanced and restrict only to

JHEP07(2004)060

E_n invariants. Once the symmetry is broken by turning on gauge interactions some of the terms in the superpotential become non-gauge invariant and are projected out. We are left then with the gauge invariant projection of the original E_n symmetric terms. In particular, the computation of superpotentials for non-toric del Pezzos has been elusive until recently. In [17], the superpotentials for some of the non-toric theories have been computed using exceptional collections. We will see along the paper how these superpotentials can also be derived by considering symmetric combinations and, in some cases, using simple inputs regarding the behavior of the theory under (un)higgsings.

As we proceed with the study of del Pezzo quivers, we will encounter a further complication. The matter content of some quivers does not even seem to fit into irreducible representations of the corresponding E_n group. We will see that it is still possible to treat all the examples within a unified framework, by introducing some new ideas such as the concept of *partial representations*. This idea simply states that the missing fields can be postulated to exist as massive fields, thus completing the representation to its actual size. Our tools allow us to identify all the quantum numbers of such missing states. A crucial ingredient in such a construction is the possibility to add a global symmetry invariant mass term for these fields such that at energies smaller than this mass such fields will be integrated out and we will be left with what appears to be a "partial representation."

The outline of this paper is as follows. In section 2, we review how E_n symmetries arise in del Pezzo surfaces, and we establish the general framework that will be used along the paper to make these symmetries explicit in the corresponding quiver theories. In section 3, we follow this methodology and, starting from $\mathbb{P}^2$, construct the divisors associated to bifundamental fields for all del Pezzo quivers up to dP_6 by performing successive blow-ups. We classify the bifundamental matter into irreducible representations of the global symmetry group and show how invariance under the global symmetry group determines superpotentials. Section 4 describes the concept of partial representations and shows in explicit examples how to determine which are the fields that are missing from them. In section 5 we identify the blow-downs that take from dP_n to dP_{n-1} with higgsing of the global symmetry group E_n down to E_{n-1} by a non-zero VEV in the fundamental representation. This offers a systematic approach to the connection among theories for different del Pezzos. Section 6 presents a simple set of rules for transforming the E_n representation content of a quiver under Seiberg duality. Sections 7 and 8 explain how to use group theory to count and classify dibaryon operators in the gauge theory, matching the geometric enumeration of such states in the AdS dual. Finally, section 9 applies the decomposition into maximal subgroups of E_n to the counting of dibaryons in dP_7 and dP_8.

2. E_n symmetries and del Pezzo surfaces

Let us have a look at how exceptional symmetries appear in del Pezzo theories.[1] Del Pezzo surfaces dP_n are manifolds of complex dimension 2 constructed by blowing up $\mathbb{P}^2$ at n generic points, $n = 0, \ldots, 8$. The lattice $H_2(dP_n, \mathbb{Z})$ is generated by the set

[1] There is a vast literature on the geometry of del Pezzo surfaces and their symmetries. Some recent papers that include nice discussions of the subject are [18, 7, 19].

$\{D, E_1, E_2, \ldots, E_n\}$. Here D is the pullback of the generator of $H_2(\mathbb{P}^2, \mathbb{Z})$ under the projection that collapses the blown-up exceptional curves $E_1, E_2, \ldots, E_n$. The intersection numbers for this basis are

$$D \cdot D = 1 \qquad D \cdot E_i = 0 \qquad E_i \cdot E_j = -\delta_{ij} \quad i, j = 1, \ldots, n \tag{2.1}$$

One can use a vector notation for the elements of $H_2(dP_n, \mathbb{Z})$ which will be useful later for counting dibaryons. In this notation the basis elements read

$$\begin{aligned} D &: \ (1, 0, 0, \ldots, 0) \\ E_1 &: \ (0, 1, 0, \ldots, 0) \\ E_2 &: \ (0, 0, 1, \ldots, 0), \ \text{etc.} \end{aligned} \tag{2.2}$$

and the intersection numbers are computed by taking the scalar product between vectors using the lorentzian metric $\mathrm{diag}(1, -1, \ldots, -1)$.

The first Chern class for dP_n is $c_1 = 3D - \sum_{i=1}^n E_i$. The canonical class is $K_n = -c_1$. The orthogonal complement of K_n according to the above intersection product is a natural sublattice of $H_2(dP_n, \mathbb{Z})$, called the *normal sublattice*. There is an isomorphism between the normal sublattice and the root lattice of the E_n Lie algebra for $n \geq 3$. If we take as basis for this sublattice the set of vectors

$$\begin{aligned} \alpha_i &= E_i - E_{i+1} \qquad i = 1, \ldots, n-1 \\ \alpha_n &= D - E_1 - E_2 - E_3 \end{aligned} \tag{2.3}$$

then the intersection numbers for the α_i are

$$\alpha_i \cdot \alpha_j = -A_{ij} \,, \qquad i, j = 1, \ldots, n \,, \tag{2.4}$$

where A_{ij} is the Cartan matrix of the Lie Algebra E_n. Thus, the α_i correspond to the simple roots of E_n. It is useful to keep in mind that $E_1 = \mathrm{U}(1)$, $E_2 = \mathrm{SU}(2) \times \mathrm{U}(1)$, $E_3 = \mathrm{SU}(2) \times \mathrm{SU}(3)$, $E_4 = \mathrm{SU}(5)$ and $E_5 = \mathrm{SO}(10)$. Given an element $\mathcal{C}$ of $H_2(dP_n, \mathbb{Z})$, we can assign a weight vector of E_n to it, with Dynkin coefficients given by its projection on the normal sublattice

$$\lambda_i = -\mathcal{C} \cdot \alpha_i \,. \tag{2.5}$$

Let us now consider the quiver gauge theories that appear on a stack of N D3-branes probing complex cones over del Pezzo surfaces.[2] For dP_n, the gauge group is $\prod_{i=1}^k \mathrm{U}(d_i N)$, where the d_i are appropriate integers and $k = n+3$, the Euler characteristic of the del Pezzo. The near horizon geometry of this configuration will be $AdS_5 \times H_5$ where H_5 is a U(1) fibration over the the del Pezzo surface dP_n. The AdS/CFT correspondence [24, 25, 26] conjectures a mapping between operators in the conformal gauge theory and states of the bulk string theory. Although this mapping is not known in its generality, it has been sufficiently explored for the special case of BPS operators. One such class of operators are

[2] Recently, the non-conformal theories resulting from the inclusion of fractional branes in these geometries and the resulting RG flows have been investigated in [20, 21, 22, 23].

dibaryons [27, 28]. These are generalizations of the usual baryons to theories with bifundamental matter. In the gravity dual dibaryons correspond to D3-branes wrapping certain 3-cycles in H_5. These 3-cycles are holomorphic 2-cycles of dP_n together with the U(1) fiber of H_5. Therefore, it is possible to assign a curve in $H_2(dP_n,\mathbb{Z})$ to every dibaryon [7, 19] (more details of this correspondence later). In the special case where the quiver theory is in the so-called toric phase,[3] i.e. when all the gauge group factors are U(N), there are some dibaryons which are formed by the anti-symmetrization of N copies of a single bifundamental field

$$\epsilon_{i_1 i_2\cdots i_N}\epsilon^{j_1 j_2\cdots j_N} X^{i_1}_{j_1} X^{i_2}_{j_2}\cdots X^{i_N}_{j_N}\,. \tag{2.6}$$

This can be repeated for every bifundamental field, allowing us to extend the correspondence between holomorphic 2-cycles (also called divisors) and dibaryons and assign an element of $H_2(dP_n,\mathbb{Z})$ to each bifundamental matter field in the quiver [7]. Thus, if $X_{\alpha\beta}$ is a bifundamental field extending from node α to node β in the quiver representation then we can associate to it an element $L_{\alpha\beta}$ of $H_2(dP_n,\mathbb{Z})$. In fact, the $L_{\alpha\beta}$ of toric quivers can be written as differences of divisors L_α associated to the nodes. The precise nature of the node divisors was clarified in [19], where they were identified with the first Chern class of the sheaves in the dual exceptional collection associated to the quiver. This result has been generalized in [19] to the case in which the ranks of the gauge groups are not necessarily equal, yielding the following expressions

$$L_{\alpha\beta} = \frac{L_\beta}{d_\beta}-\frac{L_\alpha}{d_\alpha} \qquad \text{if} \quad \frac{L_\beta}{d_\beta}-\frac{L_\alpha}{d_\alpha}\geq 0 \tag{2.7}$$

$$L_{\alpha\beta} = \frac{L_\beta}{d_\beta}-\frac{L_\alpha}{d_\alpha}+c_1 \qquad \text{if} \quad \frac{L_\beta}{d_\beta}-\frac{L_\alpha}{d_\alpha}< 0\,. \tag{2.8}$$

where the sign refers to the sign of $(L_\beta/d_\beta - L_\alpha/d_\alpha)\cdot c_1$. The supersymmetric gauge theories living on the stack of D3-branes probing these geometries are invariant under a set of global U(1) symmetries. One of these is the $\mathrm{U}(1)_R$ symmetry which is part of the superconformal algebra. There are also n flavor U(1) symmetries under which the bifundamentals are charged. Dibaryons are correspondingly charged under these symmetries, thus we refer to them as baryonic U(1)'s. The aforementioned correspondence allows us to readily calculate the charges of bifundamentals under the *baryonic* U(1)'s and $\mathrm{U}(1)_R$ contained in the global symmetry group of the quivers. In particular, the R charge, being proportional to the volume of the 3-cycle wrapped by the D3-brane in the dual geometry, is given by

$$R(X_{\alpha\beta}) = -2\frac{K_n\cdot L_{\alpha\beta}}{K_n\cdot K_n}\,. \tag{2.9}$$

The global baryonic U(1) symmetries are gauge symmetries in the AdS_5 bulk, with the U(1) gauge fields coming from the reduction of the RR gauge field C_4 on n independent 3-cycles of H_5. The flavor currents J_i of these U(1)'s must be neutral under the

[3]We want to bring to the reader's attention the particular use we are making here of the concept of a toric phase. It follows the use given to it in [29, 30] and it simply refers to a quiver in which all $d_i = 1$, i.e. all the gauge groups are equal to U(N). In particular, there can be toric quivers for non-toric del Pezzos, as we will see along the paper.

R-symmetry, which translates in the dual geometry as $J_i \cdot K_n = 0$. Therefore, the divisors J_i corresponding to these are elements of the normal sublattice and can be chosen to be the basis vectors α_i defined in (2.3). The vector of U(1) charges for each bifundamental $X_{\alpha\beta}$ is then

$$q_i = L_{\alpha\beta} \cdot J_i \,. \tag{2.10}$$

According to (2.5), these are (modulo an unimportant overall minus sign) the Dynkin coefficients of the weight vector $L_{\alpha\beta}$. We can indeed compute the weight vectors for all the toric phases of the del Pezzos (and will in fact do so in section 3). What one finds using these weight vectors is that the bifundamental matter fields can be accommodated into irreducible representations of the E_n Lie algebra for each of these theories. The matter fields within a representation have the same R charge, which is characteristic of the representation.

These theories also have superpotentials. Each term in the superpotential must be invariant under the U(1) flavor symmetries and have R-charge equal to two. The superpotential for all these models can actually be written as the gauge invariant part of singlets of the Weyl group of E_n formed by products of these irreducible representations. As we will see, this description makes it possible to recast most of what is known about the del Pezzo theories, including superpotentials, Seiberg duality relations and higgsing relations, in an elegant group theoretic language. Although the global symmetry of these models at a generic point in the moduli space is just the $\mathrm{U}(1)^n \times \mathrm{U}(1)_R$ symmetry, in the limit where all the gauge couplings $g_i \to \infty$ the full E_n symmetry is restored. This enhancement of the symmetry leaves its mark on the theory even for finite g_i, if appropriately combined with the principle of gauge invariance. In fact, the $\mathrm{U}(1)^n \times \mathrm{U}(1)_R$ global symmetry algebra forms the Cartan sub-algebra of the affine algebra, $\hat{E}_n$, with $\mathrm{U}(1)_R$ being the Cartan element associated with the imaginary root of the affine algebra. It is important to note that the sub-algebras which can be enhanced by tuning the inverse gauge coupling are always finite dimensional and, as usual, the affine algebra is never completely enhanced. It will be interesting to study the signature of the affine algebras on these quiver theories.

To conclude this section, let us stress that for non-toric quivers it is no longer possible to arrange individual bifundamental fields into representations. In fact, if we apply (2.10) to the divisors computed using (2.8), we obtain in general a set of fractional charges that cannot be interpreted as Dynkin coefficients defining a representation of the Weyl group of the corresponding E_n. It is only when various bifundamental fields are combined into dibaryons that the resulting objects have integer U(1) charges and, accordingly, well defined transformation properties under the global symmetry group. This is not surprising, since the only operators in the CFT that are mapped to the gravity side by the AdS/CFT are gauge invariant. We will discuss in section 9 how a classification of bifundamentals into subgroups of E_n is still possible and turns out to be useful for the enumeration of dibaryons.

2.1 The Weyl group and dibaryons

As mentioned above, there is an interesting relation between the Weyl group of E_n and dibaryons in del Pezzo gauge theories. Let $\mathcal{C}$ be an element of $H_2(dP_n, \mathbb{Z})$ corresponding to a dibaryon state in dP_n. The degree of this curve is defined as:

$$k = -(K_n \cdot \mathcal{C})\,. \tag{2.11}$$

Now, there is a natural action of the Weyl group of E_n on these curves that preserves their degree. If $\alpha_i \in H_2(dP_n, \mathbb{Z})$, $i = 1, \ldots, n$ is any of the simple roots in (2.3) then the corresponding Weyl group element acts on $\mathcal{C}$ as

$$w_{\alpha_i}: \quad \mathcal{C} \to \mathcal{C} + (\mathcal{C} \cdot \alpha_i)\,\alpha_i \tag{2.12}$$

and the curve produced by this action has the same degree k, because $K_n \cdot \alpha_i = 0$. Thus, the curves of a given degree form a representation of the Weyl group of E_n. So there is some E_n related structure for dibaryon states at a generic point in the moduli space, even though they do not form complete E_n representations because of the requirement of gauge invariance in their construction. On the other hand, every representation of E_n is the union of irreducible representations of the Weyl group (Weyl orbits). For basic representations, i.e. representations whose highest weight vector has only one nonzero element, equal to one, it can be shown that they consist of a single nontrivial Weyl orbit (plus n Weyl singlets in the case of the adjoint). This means that for low levels k the dimensions of Weyl orbits and irreducible E_n representations coincide (modulo a difference of n for the adjoint) and we shall see in later sections how this can be used for performing an algebraic counting of dibaryon states.

3. Global symmetry classification of quivers

The discussion in section 2 provides us with a systematic procedure to classify del Pezzo quivers according to the transformation properties of bifundamental fields under the corresponding E_n groups, which can be summarized as follows. The divisors associated to bifundamental fields are computed from the divisors assigned to the quiver nodes using (2.8). The baryonic U(1) and R charges are calculated from the intersection numbers of these divisors with the normal sublattice and the canonical class according to (2.10) and (2.9). The vector of U(1) charges for each of the matter fields in dP_n is actually a weight vector of the E_n Lie algebra and, as we will see by computing them, these weight vectors form irreducible representations of E_n.

In this section we will summarize the transformation properties under global symmetries of bifundamental fields in different phases of gauge theories on D3-branes probing complex cones over del Pezzo surfaces. This information will be used in the subsequent sections of the paper. We will closely examine the toric phases of the del Pezzo theories, dP_n for $2 \leq n \leq 6$.[4]

The corresponding superpotentials of these theories are singlets under the global symmetry transformations. Thus, they can be written as a sum of products of irreducible

[4]We skip dP_1 because its global symmetry $E_1 = \mathrm{U}(1)$ makes it rather trivial from the point of view of grouping the fields in irreducible representations (all irreducible representations are one-dimensional). For $n \leq 3$, these models have been extensively studied and the information we will use here can be found, for example, in [29]. Several aspects of the gauge theories for $4 \leq n \leq 6$ have been studied in [31, 30, 17, 7, 34].

representations, with the understanding that from each of these products we consider the E_n singlet included in it and that only the gauge invariant terms in this singlet actually contribute to the superpotential. We will only write down those superpotential terms from which some contributions survive the projection onto gauge invariants, although sometimes more terms that are invariant under global transformations can be written down.

The only subtlety we encounter in using this description is the appearance of *partial representations*. This is just a name for groups of fields that do not completely fill representations of E_n. A detailed discussion of partial representations is deferred to section 4. For now it will suffice to say that the missing components in these representations are actually massive matter fields that do not appear in the low energy limit of the theory.

In writing down these theories, we have decided to number the gauge groups following the order of the corresponding external legs of the associated (p, q) webs (we refer the reader to [31, 32] for a description of the connection between (p, q) webs and 4d gauge theories on D3-branes probing toric singularities[5]). This ordering is closely related to the one of the associated dual exceptional collection. In fact, both ordering prescriptions are almost identical, differing at most by a possible reordering of nodes within each block (set of parallel external legs in the (p, q) web description).

There are different ways to go about calculating the divisors associated to nodes and bifundamental fields in these quiver theories. According to [19], the divisors corresponding to the nodes are the elements of a dual exceptional collection, obtained through a certain braiding operation from the exceptional collection used to construct the quiver theory. Some recent progress has also been made toward reading off this dual exceptional collection from the quiver diagram of the theory [35].

We will follow here another procedure, also used in [7], which makes use of the fact that these phases are related to one another by Seiberg dualities and/or higgsing. We can compute the divisor configurations starting from any of these models and using Seiberg duality and blowing down or blowing up cycles in the geometry, which means higgsing or unhiggsing the quiver theory respectively. We will primarily focus on toric phases. By the use of Seiberg dualities on selfdual nodes (i.e. nodes whose rank does not change upon dualisation) we can 'move' among different toric phases of the theory, while the operation of blowing cycles up or down takes us from the dP_n theory to dP_{n+1} or dP_{n-1} respectively. The ways these operations act on the divisors are described in [7]. Here is a quick review of the rules for quivers in which all the gauge groups are equal to $\mathrm{U}(N)$:

- *Seiberg duality:* when a self-dual node α (i.e., for the class of quivers under consideration, a node with $2N$ flavors) is dualised, the divisor L_α changes to $L'_\alpha = L_\beta + L_\gamma - L_\alpha$ where β and γ are the nodes where arrows starting from α end or, equivalently, where arrows that end at α begin. In the case of a double arrow, β and γ can be the same.

- *Blow-down:* to blow down from dP_n to dP_{n-1} we eliminate E_n from the divisors and identify the nodes that have the same divisor after the elimination.

[5]See also [33], for a recent exploration of the correspondence.

- *Blow-up:* to go from dP_{n-1} to dP_n we add a new node and attribute to it a divisor such that the field that is unhiggsed in the quiver corresponds to the divisor E_n that we blow up. Moreover, all other divisors can differ from their blown-down counterparts only by E_n.

We now present the results of this classification for the toric phases of the del Pezzo theories, dP_n for $2 \le n \le 6$.

3.1 del Pezzo 2

The first quiver theory we examine is dP_2. The exceptional Lie algebra E_2 is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and of the two U(1) charges used below, $J_1 = E_1 - E_2$ corresponds to the Cartan generator of SU(2) and $J_2 = 2D - 3E_1 - 3E_2$ is the U(1) factor. This is a somewhat irregular case because E_2 is not semisimple. However with the choice of currents given above the fields are still organized in representations of SU(2) with each representation carrying a charge under the U(1) factor of E_2. Note that the J_2 current that is used here is different from the one used in [7]. There are two toric phases for dP_2 [29], we proceed now to study them.

Model I. This phase has 13 fields. The divisors for the nodes and the fields of this model are listed below together with the global U(1) and R charges.

$$
\begin{array}{cc}
 & \\
\text{Node} & L_\alpha \\
 & \\
1 & E_1 \\
2 & E_2 \\
3 & D \\
4 & 0 \\
5 & 2D \\
 & \\
 &
\end{array}
\qquad
\begin{array}{cccrr}
 & L_{\alpha\beta} & J_1 & J_2 & R \\
 & & & & \\
X_{41} & E_1 & -1 & 3 & 2/7 \\
X_{42} & E_2 & 1 & 3 & 2/7 \\
X_{54} & D - E_1 - E_2 & 0 & -4 & 2/7 \\
X_{52} & D - E_1 & 1 & -1 & 4/7 \\
X_{51} & D - E_2 & -1 & -1 & 4/7 \\
X_{13} & D - E_1 & 1 & -1 & 4/7 \\
X_{23} & D - E_2 & -1 & -1 & 4/7 \\
X_{35} & D & 0 & 2 & 6/7 \\
X_{34} & 2D - E_1 - E_2 & 0 & -2 & 8/7
\end{array}
\tag{3.1}
$$

The quiver diagram for this model is shown in figure 1. Throughout the paper, we will present quivers in compact block form. Every circle represents a U(N) gauge group. A block of circles at a given node of the quiver represents a set of gauge groups such that there is no bifundamental matter charged under any pair of them, and that have identical intersection numbers with the other gauge groups in the theory. The fields can be assigned to irreducible representations of SU(2) as in (3.2). The U(1) charge of each of these representations is indicated with a subscript. When the same representation appears more than once, as 2_{-1} and 1_2 do in this case, we distinguish them using a lowercase superscript (a, b, c, etc.) in parentheses, so as not to confuse it with the U(1)

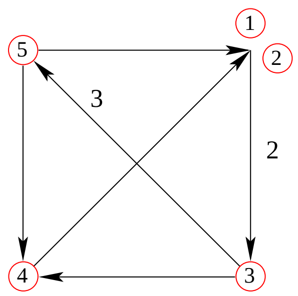

Figure 1: Quiver diagram for Model I of dP_2.

charge. This notation will be simplified for dP_n with $n \geq 3$, where no U(1) subscripts will be present. Note that all fields within a representation have the same R charge.

Fields	$\mathrm{SU}(2) \times \mathrm{U}(1)$
(X_{41}, X_{42})	2_3
(X_{52}, X_{51})	$2_{-1}^{(a)}$
(X_{13}, X_{23})	$2_{-1}^{(b)}$
(Y_{13}, Y_{23})	$2_{-1}^{(c)}$
X_{35}	$1_2^{(a)}$
Y_{35}	$1_2^{(b)}$
Z_{35}	$1_2^{(c)}$
X_{54}	1_{-4}
X_{34}	1_{-2}

(3.2)

Now we can write the superpotential of this theory as singlets of products of these representations. As discussed in section 3, only the gauge invariant terms of those singlets are actually contained in the superpotential.

$$\begin{aligned} W_I = {} & 2_3 \otimes 2_{-1}^{(a)} \otimes 1_{-2} + 2_3 \otimes 2_{-1}^{(b)} \otimes 1_{-2} + 2_3 \otimes 2_{-1}^{(c)} \otimes 1_2^{(a)} \otimes 1_{-4} + 2_3 \otimes 2_{-1}^{(c)} \otimes 1_2^{(b)} \otimes 1_{-4} + \\ & + 2_{-1}^{(a)} \otimes 2_{-1}^{(b)} \otimes 1_2^{(a)} + 2_{-1}^{(a)} \otimes 2_{-1}^{(b)} \otimes 1_2^{(b)} + 2_{-1}^{(a)} \otimes 2_{-1}^{(c)} \otimes 1_2^{(c)} \end{aligned} \tag{3.3}$$

Model II. This phase has 11 fields. It is obtained by dualising node 2 of Model I. The divisors, global charges and representation assignments are listed below.

Node	L_α
1	D
2	$2D - E_2$
3	$2D$
4	0
5	E_1

	$L_{\alpha\beta}$	J_1	J_2	R
X_{45}	E_1	-1	3	$2/7$
X_{23}	E_2	1	3	$2/7$
X_{34}	$D - E_1 - E_2$	0	-4	$2/7$
X_{24}	$D - E_1$	1	-1	$4/7$
X_{35}	$D - E_2$	-1	-1	$4/7$
X_{51}	$D - E_1$	1	-1	$4/7$
X_{12}	$D - E_2$	-1	-1	$4/7$
X_{41}	D	0	2	$6/7$
X_{13}	D	0	2	$6/7$

(3.4)

Figure 2 shows the quiver for this model. There are two double arrows in this theory, given by the fields X_{12}, Y_{12} X_{51} and Y_{51}. Identical divisors are assigned to fields connecting the same pair of nodes. In (3.4), we list only one field for each double arrow. This convention will be followed for all other models with multiple arrows in the paper. As above, SU(2) representations and U(1) charges are presented in table (3.5).

JHEP07(2004)060

Fields	$SU(2) \times U(1)$
(X_{23}, X_{45})	$\mathbf{2}_3$
(X_{35}, X_{24})	$\mathbf{2}_{-1}^{(a)}$
(X_{12}, X_{51})	$\mathbf{2}_{-1}^{(b)}$
(Y_{12}, Y_{51})	$\mathbf{2}_{-1}^{(c)}$
X_{13}	$\mathbf{1}_2^{(a)}$
X_{41}	$\mathbf{1}_2^{(b)}$
X_{34}	$\mathbf{1}_{-4}$

(3.5)

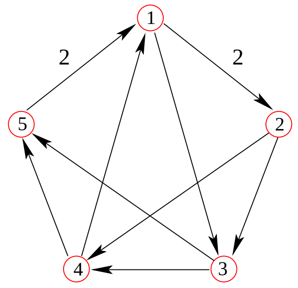

Figure 2: Quiver diagram for Model II of dP_2.

The superpotential for this theory becomes

$$\begin{aligned} W_{II} = \; & \mathbf{1}_2^{(a)} \otimes \mathbf{1}_{-4} \otimes \mathbf{1}_2^{(b)} + \mathbf{1}_2^{(b)} \otimes \mathbf{2}_{-1}^{(c)} \otimes \mathbf{2}_{-1}^{(a)} + \mathbf{1}_2^{(a)} \otimes \mathbf{2}_{-1}^{(a)} \otimes \mathbf{2}_{-1}^{(c)} + \\ & + \mathbf{2}_{-1}^{(c)} \otimes \mathbf{2}_{-1}^{(b)} \otimes \mathbf{2}_{-1}^{(a)} \otimes \mathbf{2}_3 + \mathbf{2}_{-1}^{(b)} \otimes \mathbf{2}_{-1}^{(b)} \otimes \mathbf{2}_3 \otimes \mathbf{1}_{-4} \otimes \mathbf{2}_3 \, . \end{aligned} \tag{3.6}$$

The terms $\mathbf{1}_2^{(a)} \otimes \mathbf{1}_{-4} \otimes \mathbf{2}_3 \otimes \mathbf{2}_{-1}^{(b)}$ and $\mathbf{1}_2^{(a)} \otimes \mathbf{1}_{-4} \otimes \mathbf{2}_3 \otimes \mathbf{2}_{-1}^{(c)}$ are globally symmetric and gauge invariant, but nevertheless are not present in the superpotential. We can check that this is so by looking at how it is generated by higgsing dP_3.

3.2 del Pezzo 3

There are four toric phases for dP_3, related to one another by Seiberg dualities [29]. For each of them we list the divisors corresponding to the nodes and fields, the assignment of fields to representations and the superpotential written as an E_3 singlet. We obtain dP_3 by blowing up a 2-cycle in dP_2.

Model I. This model has 12 fields.

Node	L_α
1	$D + E_3$
2	$2D - E_2$
3	$2D$
4	E_3
5	$E_1 + E_3$
6	D

	$L_{\alpha\beta}$	J_1	J_2	J_3	R
X_{45}	E_1	-1	0	1	$1/3$
X_{23}	E_2	1	-1	1	$1/3$
X_{61}	E_3	0	1	1	$1/3$
X_{34}	$D - E_1 - E_2$	0	1	-1	$1/3$
X_{56}	$D - E_1 - E_3$	1	-1	-1	$1/3$
X_{12}	$D - E_2 - E_3$	-1	0	-1	$1/3$
X_{51}	$D - E_1$	1	0	0	$2/3$
X_{35}	$D - E_2$	-1	1	0	$2/3$
X_{13}	$D - E_3$	0	-1	0	$2/3$
X_{24}	$D - E_1$	1	0	0	$2/3$
X_{62}	$D - E_2$	-1	1	0	$2/3$
X_{46}	$D - E_3$	0	-1	0	$2/3$

(3.7)

The quiver diagram for this model is displayed in figure 3. The fields are arranged in irreducible representations of $E_3 = SU(2) \times SU(3)$ as shown in (3.8). The assignment is done by comparing the U(1) charges above with the Dynkin labels of the weight vectors of an E_3 irreducible representation. We remind the reader that because of the difference in their definitions there is an overall minus sign difference that must be taken into account in this comparison. From now on, repeated representations will be identified with lowercase subscript (a, b, c, etc.).

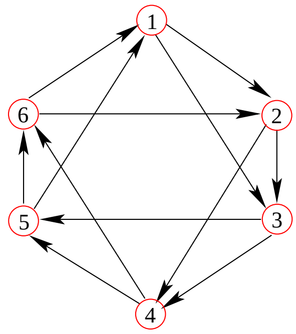

Figure 3: Quiver diagram for Model I of dP_3.

Fields	$SU(2) \times SU(3)$
$(X_{12}, X_{23}, X_{34}, X_{45}, X_{56}, X_{61})$	$(2, 3)$
(X_{13}, X_{35}, X_{51})	$(1, \bar{3})_a$
(X_{24}, X_{46}, X_{62})	$(1, \bar{3})_b$

(3.8)

The superpotential can then be written as

$$W_I = (2,3)^6 + (2,3)^2 \otimes (1,\bar{3})_a \otimes (1,\bar{3})_b + (1,\bar{3})_a^3 + (1,\bar{3})_b^3 \,. \tag{3.9}$$

Model II. There are 14 fields in this phase. We can get it by dualising node 1 of the previous model. Again, we calculate the divisors and charges and compare them with E_3 weight vectors in order to assign the fields to representations. The results are shown in the tables below.

Node	L_α		$L_{\alpha\beta}$	J_1	J_2	J_3	R
		X_{32}	E_1	-1	0	1	$1/3$
		X_{65}	E_2	1	-1	1	$1/3$
Node	L_α	X_{42}	E_3	0	1	1	$1/3$
		X_{53}	$D - E_1 - E_2$	0	1	-1	$1/3$
1	D	X_{21}	$D - E_1 - E_3$	1	-1	-1	$1/3$
2	$E_1 + E_3$	X_{54}	$D - E_2 - E_3$	-1	0	-1	$1/3$
3	E_3	X_{41}	$D - E_1$	1	0	0	$2/3$
4	E_1	X_{16}	$D - E_2$	-1	1	0	$2/3$
5	$2D$	X_{64}	$D - E_3$	0	-1	0	$2/3$
6	$2D - E_2$	X_{63}	$D - E_1$	1	0	0	$2/3$
		Y_{16}	$D - E_2$	-1	1	0	$2/3$
		X_{31}	$D - E_3$	0	-1	0	$2/3$
		X_{15}	D	0	0	1	1
		X_{26}	$2D - E_1 - E_2 - E_3$	0	0	-1	1

(3.10)

The quiver for this model is shown in figure 4. The representation structure is shown in (3.11).

Fields	SU(2) × SU(3)
$(X_{54}, X_{65}, X_{53}, X_{32}, X_{21}, X_{42})$	$(2,3)$
(X_{63}, X_{31}, X_{16})	$(1,\bar{3})_a$
(X_{64}, X_{41}, Y_{16})	$(1,\bar{3})_b$
(X_{26}, X_{15})	$(2,1)$

(3.11)

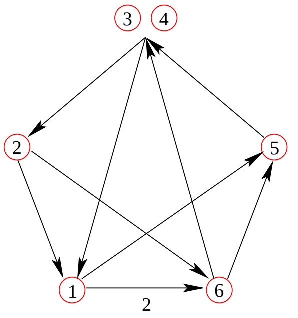

Figure 4: Quiver diagram for Model II of dP_3.

Using this, we can write the superpotential as

$$\begin{aligned} W_{II} = {} & (2,3)^4 \otimes (1,\bar{3})_a + (2,3)^4 \otimes (1,\bar{3})_b + \\ & +(2,3) \otimes (2,1) \otimes (1,\bar{3})_a + (2,3) \otimes (2,1) \otimes (1,\bar{3})_b + \\ & +(1,\bar{3})_a^2 \otimes (1,\bar{3})_b + (1,\bar{3})_a \otimes (1,\bar{3})_b^2 \,. \end{aligned} \tag{3.12}$$

Model III. Dualising node 2 of Model II we obtain Model III. This phase has 14 fields.

Node	L_α
1	D
2	E_1
3	E_3
4	0
5	$2D$
6	$2D - E_2$

	$L_{\alpha\beta}$	J_1	J_2	J_3	R
X_{42}	E_1	-1	0	1	1/3
X_{65}	E_2	1	-1	1	1/3
X_{43}	E_3	0	1	1	1/3
X_{53}	$D - E_1 - E_2$	0	1	-1	1/3
X_{64}	$D - E_1 - E_3$	1	-1	-1	1/3
X_{52}	$D - E_2 - E_3$	-1	0	-1	1/3
X_{21}	$D - E_1$	1	0	0	2/3
X_{16}	$D - E_2$	-1	1	0	2/3
X_{31}	$D - E_3$	0	-1	0	2/3
X_{15}	D	0	0	1	1
X_{14}	$2D - E_1 - E_2 - E_3$	0	0	-1	1

(3.13)

Figure 5 shows the quiver diagram and table (3.14) summarizes how the fields fall into representations.

Fields	SU(2) × SU(3)
$(X_{21}, X_{64}, X_{23}, X_{43}, X_{62}, X_{41})$	$(2,3)$
(X_{15}, X_{56}, X_{35})	$(1,\bar{3})_a$
(Y_{15}, Y_{56}, Y_{35})	$(1,\bar{3})_b$
(X_{54}, X_{52})	$(2,1)$

(3.14)

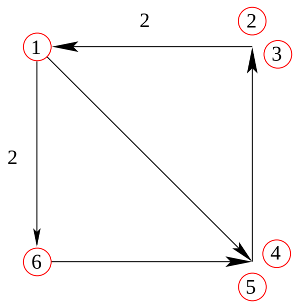

Figure 5: Quiver diagram for Model III of dP_3.

The superpotential for this theory can be written in an invariant form as

$$\begin{aligned} W_{III} = {} & (2,3)^2 \otimes (1,\bar{3})_a^2 + (2,3)^2 \otimes (1,\bar{3})_a \otimes (1,\bar{3})_b + (2,3)^2 \otimes (1,\bar{3})_b^2 + \\ & +(2,3) \otimes (2,1) \otimes (1,\bar{3})_a + (2,3) \otimes (2,1) \otimes (1,\bar{3})_b \,. \end{aligned} \tag{3.15}$$

JHEP07(2004)060

Model IV. There are 18 fields in this phase, which is produced by dualising node 6 of Model III.

Node	L_α
1	E_1
2	E_2
3	E_3
4	$2D$
5	0
6	D

	$L_{\alpha\beta}$	J_1	J_2	J_3	R
X_{51}	E_1	-1	0	1	$1/3$
X_{52}	E_2	1	-1	1	$1/3$
X_{53}	E_3	0	1	1	$1/3$
X_{43}	$D-E_1-E_2$	0	1	-1	$1/3$
X_{42}	$D-E_1-E_3$	1	-1	-1	$1/3$
X_{41}	$D-E_2-E_3$	-1	0	-1	$1/3$
X_{16}	$D-E_1$	1	0	0	$2/3$
X_{26}	$D-E_2$	-1	1	0	$2/3$
X_{36}	$D-E_3$	0	-1	0	$2/3$
X_{64}	D	0	0	1	1
X_{65}	$2D-E_1-E_2-E_3$	0	0	-1	1

(3.16)

The table in (3.17) shows how the fields are organized in representations.

Fields	$\mathrm{SU}(2)\times\mathrm{SU}(3)$
$(X_{51}, X_{52}, X_{53}, X_{41}, X_{42}, X_{43})$	$(2,3)$
(X_{16}, X_{26}, X_{36})	$(1,\bar{3})_a$
(Y_{16}, Y_{26}, Y_{36})	$(1,\bar{3})_b$
(X_{64}, X_{65})	$(2,1)_a$
(Y_{64}, Y_{65})	$(2,1)_b$
(Z_{64}, Z_{65})	$(2,1)_c$

(3.17)

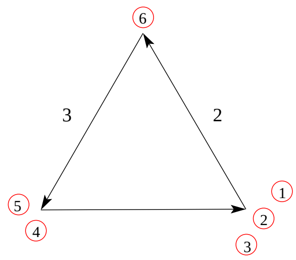

Figure 6: Quiver diagram for Model IV of dP_3.

The superpotential is

$$W_{IV} = (2,3) \otimes [(1,\bar{3})_a + (1,\bar{3})_b] \otimes [(2,1)_a + (2,1)_b + (2,1)_c]\,. \tag{3.18}$$

3.3 del Pezzo 4

There are two toric phases for dP_4. The organization of matter fields into E_4 representations is in this case more subtle than in the preceding examples. The classification of these theories can be achieved with the same reasoning as before, by introducing the idea of partial representations. Partial representations are ordinary representations in which some of the fields are massive, being integrated out in the low energy limit. We will summarize the results in this section, and postpone a detailed explanation of partial representations to sections 4 and 6.

Model I. This theory has 15 fields.

Node	L_α
1	D
2	E_1
3	E_3
4	$2D-E_4$
5	0
6	$D-E_4$
7	$2D-E_2-E_4$

	$L_{\alpha\beta}$	J_1	J_2	J_3	J_4	R
X_{45}	E_1	-1	0	0	1	$2/5$
X_{23}	E_2	1	-1	0	1	$2/5$
X_{46}	E_3	0	1	-1	1	$2/5$
X_{71}	E_4	0	0	1	0	$2/5$
X_{36}	$D-E_1-E_2$	0	1	0	-1	$2/5$
X_{24}	$D-E_1-E_3$	1	-1	1	-1	$2/5$
X_{57}	$D-E_1-E_4$	1	0	-1	0	$2/5$
X_{35}	$D-E_2-E_3$	-1	0	1	-1	$2/5$
X_{12}	$D-E_2-E_4$	-1	1	-1	0	$2/5$
X_{67}	$D-E_3-E_4$	0	-1	0	0	$2/5$
X_{51}	$D-E_1$	1	0	0	0	$4/5$
X_{72}	$D-E_2$	-1	1	0	0	$4/5$
X_{61}	$D-E_3$	0	-1	1	0	$4/5$
X_{13}	$D-E_4$	0	0	-1	1	$4/5$
X_{14}	$2D-E_1-E_2-E_3-E_4$	0	0	0	-1	$4/5$

(3.19)

Comparing the charges with weight vectors of SU(5) representations we find the assignment tabulated in (3.20).

Fields	SU(5)
$(X_{45}, X_{23}, X_{46}, X_{71}, X_{36}, X_{24}, X_{57}, X_{35}, X_{12}, X_{67})$	10
$(X_{51}, X_{72}, X_{67}, X_{13}, X_{14})$	$\bar{5}$

(3.20)

The superpotential is written in terms of singlets as

$$W_I = 10 \otimes \bar{5}^2 + 10^3 \otimes \bar{5}\,. \tag{3.21}$$

One might wonder whether a $\mathbf{10}^5$ term should be present in $W_{dP_{4,I}}$. At first sight it appears as a valid contribution, since this product of representations contains an E_4 singlet and we see from figure 7 that it would survive the projection onto gauge invariant states. As we shall discuss in section 5, this model can be obtained from Model I of dP_5 by higgsing. All the gauge invariants in that theory are quartic. In particular, since there are no cubic terms, masses are not generated when turning on a non-zero vev for a bifundamental field. Then, we conclude that any fifth order term in $W_{dP_{4,I}}$ should have its origin either in a fifth or sixth order term in $W_{dP_{5,I}}$. Since $W_{dP_{5,I}}$ is purely quartic, we conclude that the $\mathbf{10}^5$ is not present in $W_{dP_{4,I}}$.

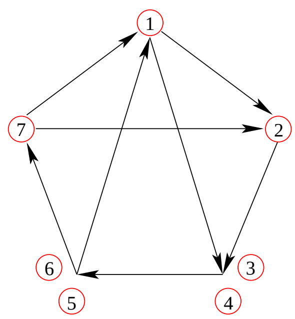

Figure 7: Quiver diagram for Model I of dP_4.

Model II. Upon dualisation of node 7 of Model I we get Model II. There are 19 fields in this model.

Node	L_α
1	D
2	E_1
3	E_3
4	$2D - E_4$
5	0
6	$2D - E_2$
7	$2D - E_2 - E_4$

	$L_{\alpha\beta}$	J_1	J_2	J_3	J_4	R
X_{52}	E_1	-1	0	0	1	$2/5$
X_{74}	E_2	1	-1	0	1	$2/5$
X_{53}	E_3	0	1	-1	1	$2/5$
X_{76}	E_4	0	0	1	0	$2/5$
X_{43}	$D - E_1 - E_2$	0	1	0	-1	$2/5$
X_{75}	$D - E_1 - E_3$	1	-1	1	-1	$2/5$
X_{63}	$D - E_1 - E_4$	1	0	-1	0	$2/5$
X_{42}	$D - E_2 - E_3$	-1	0	1	-1	$2/5$
X_{17}	$D - E_2 - E_4$	-1	1	-1	0	$2/5$
X_{62}	$D - E_3 - E_4$	0	-1	0	0	$2/5$
X_{21}	$D - E_1$	1	0	0	0	$4/5$
X_{16}	$D - E_2$	-1	1	0	0	$4/5$
X_{31}	$D - E_3$	0	-1	1	0	$4/5$
X_{14}	$D - E_4$	0	0	-1	1	$4/5$
X_{15}	$2D - E_1 - E_2 - E_3 - E_4$	0	0	0	-1	$4/5$
X_{27}	$2D - E_1 - E_2 - E_4$	0	1	-1	0	$6/5$
X_{37}	$2D - E_2 - E_3 - E_4$	-1	0	0	0	$6/5$

(3.22)

This is the first example where partial representations appear. We will study this further in section 4, and for the moment it will suffice to say that the missing fields (indicated by asterisks in table (3.23)) are massive and the terms containing those fields in the singlets forming the superpotential should be thrown out, as they are not a part of the low energy theory.

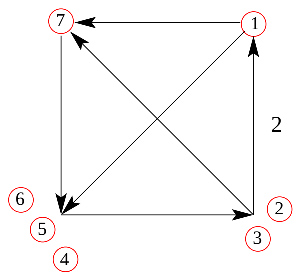

Figure 8: Quiver diagram for Model II of dP_4.

Fields	SU(5)
$(X_{52}, X_{74}, X_{53}, X_{76}, X_{43}, X_{75}, X_{63}, X_{42}, X_{17}, X_{62})$	10
$(X_{21}, X_{16}, X_{31}, X_{14}, X_{15})$	$\bar{5}_a$
$(Y_{21}, Y_{31}, *, *, *)$	partial $\bar{5}_b$
$(X_{27}, X_{37}, *, *, *)$	partial 5

(3.23)

The superpotential for this theory is

$$W_{II} = 10 \otimes \bar{5}_a^2 + 10 \otimes \bar{5}_a \otimes \bar{5}_b + 10^3 \otimes \bar{5}_a + 10^3 \otimes \bar{5}_b + 10^2 \otimes 5 \tag{3.24}$$

where the superpotential corresponds only to those E_4 invariant contributions that survive the projection onto gauge invariant terms. The terms including the partial $\bar{\mathbf{5}}_{\mathbf{b}}$ representation are naturally understood to be truncated to fields actually appearing in the quiver.

JHEP07(2004)060

3.4 del Pezzo 5

Let us study the three toric phases of dP_5. Models II and III exhibit again the phenomenon of partial representations.

Model I. This model has 16 fields.

Node	L_α
1	E_1
2	E_3
3	$D-E_4$
4	$D-E_5$
5	$2D-E_2-E_4-E_5$
6	D
7	$2D-E_4-E_5$
8	0

	$L_{\alpha\beta}$	J_1	J_2	J_3	J_4	J_5	R
X_{81}	E_1	-1	0	0	0	1	$1/2$
X_{57}	E_2	1	-1	0	0	1	$1/2$
X_{82}	E_3	0	1	-1	0	1	$1/2$
X_{36}	E_4	0	0	1	-1	0	$1/2$
X_{46}	E_5	0	0	0	1	0	$1/2$
X_{72}	$D-E_1-E_2$	0	1	0	0	-1	$1/2$
X_{58}	$D-E_1-E_3$	1	-1	1	0	-1	$1/2$
X_{13}	$D-E_1-E_4$	1	0	-1	1	0	$1/2$
X_{14}	$D-E_1-E_5$	1	0	0	-1	0	$1/2$
X_{71}	$D-E_2-E_3$	-1	0	1	0	-1	$1/2$
X_{45}	$D-E_2-E_4$	-1	1	-1	1	0	$1/2$
X_{35}	$D-E_2-E_5$	-1	1	0	-1	0	$1/2$
X_{23}	$D-E_3-E_4$	0	-1	0	1	0	$1/2$
X_{24}	$D-E_3-E_5$	0	-1	1	-1	0	$1/2$
X_{67}	$D-E_4-E_5$	0	0	-1	0	1	$1/2$
X_{68}	$2D-E_1-E_2-E_3-E_4-E_5$	0	0	0	0	-1	$1/2$

(3.25)

All sixteen fields here are accommodated in a single **16** representation of $E_5 = \mathrm{SO}(10)$.

Fields	SO(10)
$(X_{81}, X_{57}, X_{82}, X_{36}, X_{46}, X_{72}, X_{58}, X_{13}, X_{14}, X_{71}, X_{45}, X_{35}, X_{23}, X_{24}, X_{67}, X_{68})$	16

(3.26)

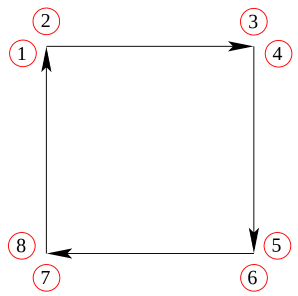

Figure 9: Quiver diagram for Model I of dP_5.

The superpotential consists simply of all quartic gauge invariants and can be written as

$$W_I = 16^4 \,. \tag{3.27}$$

Model II. Model II is can be obtained by dualising node 5 of model I. The divisors and charges are as shown in the tables.

Node	L_α
1	D
2	$2D - E_4 - E_5$
3	0
4	E_1
5	E_2
6	E_3
7	$D - E_4$
8	$D - E_5$

	$L_{\alpha\beta}$	J_1	J_2	J_3	J_4	J_5	R
X_{34}	E_1	-1	0	0	0	1	$1/2$
X_{35}	E_2	1	-1	0	0	1	$1/2$
X_{36}	E_3	0	1	-1	0	1	$1/2$
X_{71}	E_4	0	0	1	-1	0	$1/2$
X_{81}	E_5	0	0	0	1	0	$1/2$
X_{26}	$D - E_1 - E_2$	0	1	0	0	-1	$1/2$
X_{25}	$D - E_1 - E_3$	1	-1	1	0	-1	$1/2$
X_{47}	$D - E_1 - E_4$	1	0	-1	1	0	$1/2$
X_{48}	$D - E_1 - E_5$	1	0	0	-1	0	$1/2$
X_{24}	$D - E_2 - E_3$	-1	0	1	0	-1	$1/2$
X_{57}	$D - E_2 - E_4$	-1	1	-1	1	0	$1/2$
X_{58}	$D - E_2 - E_5$	-1	1	0	-1	0	$1/2$
X_{67}	$D - E_3 - E_4$	0	-1	0	1	0	$1/2$
X_{68}	$D - E_3 - E_5$	0	-1	1	-1	0	$1/2$
X_{12}	$D - E_4 - E_5$	0	0	-1	0	1	$1/2$
X_{13}	$2D - E_1 - E_2 - E_3 - E_4 - E_5$	0	0	0	0	-1	$1/2$
X_{82}	$D - E_4$	0	0	-1	1	1	1
X_{72}	$D - E_5$	0	0	0	-1	1	1
X_{83}	$2D - E_1 - E_2 - E_3 - E_4$	0	0	0	1	-1	1
X_{73}	$2D - E_1 - E_2 - E_3 - E_5$	0	0	1	-1	-1	1

(3.28)

The meson fields created by the dualization form a partial **10** representation of SO(10).

Fields	SO(10)
$(X_{34}, X_{35}, X_{36}, X_{71}, X_{81}, X_{26}, X_{25}, X_{47}, X_{48}, X_{24}, X_{57}, X_{58}, X_{67}, X_{68}, X_{12}, X_{13})$	16
$(X_{82}, X_{72}, X_{83}, X_{73}, *, *, *, *, *, *)$	partial 10

(3.29)

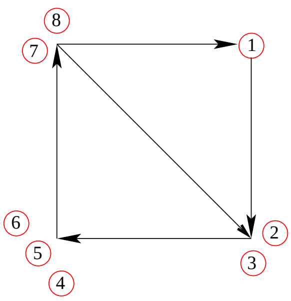

Figure 10: Quiver diagram for Model II of dP_5.

The superpotential is

$$W_{II} = 16^4 + 16^2 \otimes 10 \,. \tag{3.30}$$

Model III. The last toric phase of dP_5 is produced by dualising on node 6 of model II and has 24 fields, arranged as follows.

Node	L_α
1	E_1
2	E_3
3	E_2
4	$D-E_4-E_5$
5	$2D-E_4-E_5$
6	0
7	$D-E_4$
8	$D-E_5$

	$L_{\alpha\beta}$	J_1	J_2	J_3	J_4	J_5	R
X_{61}	E_1	-1	0	0	0	1	$1/2$
X_{63}	E_2	1	-1	0	0	1	$1/2$
X_{62}	E_3	0	1	-1	0	1	$1/2$
X_{48}	E_4	0	0	1	-1	0	$1/2$
X_{47}	E_5	0	0	0	1	0	$1/2$
X_{52}	$D-E_1-E_2$	0	1	0	0	-1	$1/2$
X_{53}	$D-E_1-E_3$	1	-1	1	0	-1	$1/2$
X_{17}	$D-E_1-E_4$	1	0	-1	1	0	$1/2$
X_{18}	$D-E_1-E_5$	1	0	0	-1	0	$1/2$
X_{51}	$D-E_2-E_3$	-1	0	1	0	-1	$1/2$
X_{37}	$D-E_2-E_4$	-1	1	-1	1	0	$1/2$
X_{38}	$D-E_2-E_5$	-1	1	0	-1	0	$1/2$
X_{27}	$D-E_3-E_4$	0	-1	0	1	0	$1/2$
X_{28}	$D-E_3-E_5$	0	-1	1	-1	0	$1/2$
X_{64}	$D-E_4-E_5$	0	0	-1	0	1	$1/2$
X_{54}	$2D-E_1-E_2-E_3-E_4-E_5$	0	0	0	0	-1	$1/2$
X_{85}	$D-E_4$	0	0	-1	1	1	1
X_{75}	$D-E_5$	0	0	0	-1	1	1
X_{86}	$2D-E_1-E_2-E_3-E_4$	0	0	0	1	-1	1
X_{76}	$2D-E_1-E_2-E_3-E_5$	0	0	1	-1	-1	1

(3.31)

A second copy of the partial **10** representation appears here.

Fields	SO(10)
$(X_{61}, X_{63}, X_{62}, X_{48}, X_{47}, X_{52}, X_{53}, X_{17}, X_{18}, X_{51}, X_{37}, X_{38}, X_{27}, X_{28}, X_{64}, X_{54})$	16
$(X_{85}, X_{75}, X_{86}, X_{76}, *, *, *, *, *, *)$	partial 10_a
$(Y_{85}, Y_{75}, Y_{86}, Y_{76}, *, *, *, *, *, *)$	partial 10_b

(3.32)

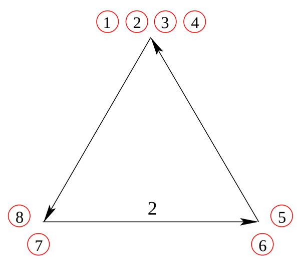

Figure 11: Quiver diagram for Model III of dP_5.

The superpotential for this theory is

$$W_{III} = 16^2 \otimes 10_a + 16^2 \otimes 10_b\,. \tag{3.33}$$

Note that the global symmetry invariant term $\mathbf{16}^4$ does not appear, since none of its components survives the projection onto gauge invariants. It is clear from looking at the 3-block quiver in figure 11 that there are no quartic gauge invariants in this case.

3.5 del Pezzo 6

The final model we study is the toric phase of dP_6. There are some indications suggesting that this theory completes the list of toric phases of del Pezzo theories. In particular, the geometric computation of dibaryon R charges (the charges of bifundamental fields can be derived from them) determines that, if all gauge groups were $\mathrm{U}(N)$, the least possible R-charge for a bifundamental field is one for dP_7 and two for dP_8. In this case, the superpotential for dP_7 could only consist of quadratic mass terms, while it would be impossible to construct a superpotential for dP_8 [34]. Since we expect all del Pezzo theories to have nontrivial superpotentials, this seems to rule out such models. Some other particular features, that might be related to the previous one, and differentiate dP_7 and dP_8 from the rest of the del Pezzos arise in the context of (p, q) webs [31, 32], where webs without crossing external legs cannot be constructed beyond dP_6.

The toric phase of dP_6 has 27 fields. The associated divisors are listed in (3.34).

Node	L_α
1	E_1
2	E_2
3	E_3
4	$D-E_4$
5	$D-E_5$
6	$D-E_6$
7	D
8	0
9	$2D-E_4-E_5-E_6$

	$L_{\alpha\beta}$	J_1	J_2	J_3	J_4	J_5	J_6	R
X_{81}	E_1	-1	0	0	0	0	1	2/3
X_{82}	E_2	1	-1	0	0	0	1	2/3
X_{83}	E_3	0	1	-1	0	0	1	2/3
X_{47}	E_4	0	0	1	-1	0	0	2/3
X_{57}	E_5	0	0	0	1	-1	0	2/3
X_{67}	E_6	0	0	0	0	1	0	2/3
X_{93}	$D-E_1-E_2$	0	1	0	0	0	-1	2/3
X_{92}	$D-E_1-E_3$	1	-1	1	0	0	-1	2/3
X_{14}	$D-E_1-E_4$	1	0	-1	1	0	0	2/3
X_{15}	$D-E_1-E_5$	1	0	0	-1	1	0	2/3
X_{16}	$D-E_1-E_6$	1	0	0	0	-1	0	2/3
X_{91}	$D-E_2-E_3$	-1	0	1	0	0	-1	2/3
X_{24}	$D-E_2-E_4$	-1	1	-1	1	0	0	2/3
X_{25}	$D-E_2-E_5$	-1	1	0	-1	1	0	2/3
X_{26}	$D-E_2-E_6$	-1	1	0	0	-1	0	2/3
X_{34}	$D-E_3-E_4$	0	-1	0	1	0	0	2/3
X_{35}	$D-E_3-E_5$	0	-1	1	-1	1	0	2/3
X_{36}	$D-E_3-E_6$	0	-1	1	0	-1	0	2/3
X_{69}	$D-E_4-E_5$	0	0	-1	0	1	1	2/3
X_{59}	$D-E_4-E_6$	0	0	-1	1	-1	1	2/3
X_{49}	$D-E_5-E_6$	0	0	0	-1	0	1	2/3
X_{71}	$2D-E_2-E_3-E_4-E_5-E_6$	-1	0	0	0	0	0	2/3
X_{72}	$2D-E_1-E_3-E_4-E_5-E_6$	1	-1	0	0	0	0	2/3
X_{73}	$2D-E_1-E_2-E_4-E_5-E_6$	0	1	-1	0	0	0	2/3
X_{48}	$2D-E_1-E_2-E_3-E_5-E_6$	0	0	1	-1	0	-1	2/3
X_{58}	$2D-E_1-E_2-E_3-E_4-E_6$	0	0	0	1	-1	-1	2/3
X_{68}	$2D-E_1-E_2-E_3-E_4-E_5$	0	0	0	0	1	-1	2/3

(3.34)

All fields are accommodated in the fundamental **27** representation of E_6, as shown in (3.36). The superpotential for this theory is simply

$$W = 27^3\,. \tag{3.35}$$

Fields	E_6
$(X_{81}, X_{82}, X_{83}, X_{47}, X_{57}, X_{67}, X_{93}, X_{92}, X_{14}$ $X_{15}, X_{16}, X_{91}, X_{24}, X_{25}, X_{26}, X_{34}, X_{35}, X_{36}$ $X_{69}, X_{59}, X_{49}, X_{71}, X_{72}, X_{73}, X_{48}, X_{58}, X_{68})$	27

(3.36)

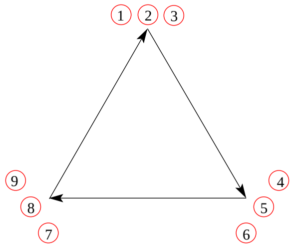

Figure 12: Quiver diagram for dP_6.

4. Partial represention

We have discussed how the matter content of the quiver theories for each dP_n can be arranged into irreducible representations of the corresponding E_n group. The superpotential can therefore be expressed as the gauge invariant part of a combination of fields invariant under the global symmetry transformations. Our discussion of this classification will be extended in section 6, where we will study how to relate the representation content of Seiberg dual theories.

In section 3, we encoutered some examples (Model II of dP_4, and Models II and III of dP_5) that seem to challenge the applicability of our classification strategy. We mentioned there that in these cases we have to go a step further and consider *partial representations*, and postponed the explanation to this point. The purpose of this section is to give a detailed description of the concept of partial representations and to show that they are a natural construction that enables us to study all the toric del Pezzo quivers from the same unified perspective. We will devote this subsection to explaining the simple rules that can be derived for these theories from a field theory point of view. The next subsection will sharpen these concepts but bases the discussion on the geometry of partial representations.

Theories with *partial representations* are those in which it is not possible to arrange matter fields so that the corresponding E_n representations are completely filled. Naively, it is not clear what are the transformation properties that should be assigned to fields that seem not to fit into representations in these cases. It is not even clear that they can be organized into irreducible representations at all. As we will discuss, this situation neither implies a loss of predictive power nor that these models are exceptionl cases outside of the scope of our techniques, since in order for partial representations to exist, very specific conditions have to be fulfilled.

The idea is to find those fields that seem to be absent from the quiver, and that would join the fields that are present to form irreducible E_n representations. These bifundamental fields should appear in representations and have gauge charges such that, following the rules given in section 3, quadratic terms appear in the superpotential. That is they can form quadratic invariants of the global symmetry group and, in quiver language, they appear as bidirectional arrows. These symmetric terms give masses to the fields under consideration, removing them from the low energy effective description.

Following the previous reasoning, partial representations appear in such a way that the same number of fields are missing from those representations that form quadratic terms. In some cases it is possible for the missing fields to lie on the same representation, which combines with itself to form a quadratic invariant. When this occurs, the number of missing fields is even. The R charges of fields in representations that combine into quadratic invariants and become partial representations add up to 2. Thus, for the specific case of self-combining representations, they should have $R = 1$ in order to be capable of becoming partial.

These general concepts are sufficient to classify the quivers into representations, but do not indicate which are the precise nodes that are connected by the missing fields. One possible way to determine them uses the assignation of divisors to bifundamental fields and is the motive of the next subsection.

4.1 The geometry of partial representations

An important question is what the location in the quiver of the fields that are needed in order to complete partial representations is. As we discussed in the previous section, fields missing from partial representations form bidirectional arrows and are combined into quadratic mass terms.

The lists in section 3 summarize the baryonic U(1) and R charges of the fields that are present in the quiver. For each arrow, these numbers indicate the intersection of its associated divisor with the $n+1$ curves in the non-orthogonal basis of (2.3). Thus, these charges define a set of $n+1$ equations from which the divisor associated to a given bifundamental field can be deduced. Furthermore, as explained in section 2, the n flavor charges correspond to the Dynkin components of each state. Then, the Dynkin components of the missing fields can be inferred by looking for those that are absent from partial representations. Once these charges are determined, they can be used together with the R charge of the representation to follow the process explained above and establish the divisors for the missing fields.

Based on the divisors that correspond to each node, (2.8) gives the divisors for every possible bifundamental field. Comparing them with the ones for missing fields we determine where they are in the quiver. Let us remark that the examples in section 3 show us that different bifundamentals can have the same associated divisors. In the cases we will study, it is straightforward to check that such ambiguity does not hold for the fields we are trying to identify.

Let us consider the example of Model II of dP_4. There are 19 fields in this theory. Some of them form a full $\mathbf{10}$ and a full $\bar{\mathbf{5}}$ representations. There are four remaining fields that cannot be arranged into full representations of E_4. From the Dynkin components in (3.22), we conclude that X_{27} and X_{37} sit in an incomplete $\mathbf{5}$, while Y_{21} and Y_{31} are part of a $\bar{\mathbf{5}}$. There are six missing fields that should complete the $\mathbf{5}$ and $\bar{\mathbf{5}}$. The Dynkin components (U(1) charges) that are needed to complete the representations can be immediately determined. They are listed in the second column of (4.1). From those, the divisors in the third column are computed. The divisors for each node in the quiver appear in (3.22). Using them, we determine the nodes in the quiver that are connected by the missing fields.

Representation	(J_1, J_2, J_3, J_4, R)	Divisor	Bifundamental
$\bar{5}$	$(-1, 1, 0, 0, 4/5)$	$D - E_2$	Y_{16}
	$(0, 0, -1, 1, 4/5)$	$D - E_4$	Y_{14}
	$(0, 0, 0, -1, 4/5)$	$2D - E_1 - E_2 - E_3 - E_4$	Y_{15}
5	$(1, -1, 0, 0, 6/5)$	$2D - E_1 - E_3 - E_4$	X_{61}
	$(0, 0, 1, -1, 6/5)$	$2D - E_1 - E_2 - E_3$	X_{41}
	$(0, 0, 0, 1, 6/5)$	D	X_{51}

(4.1)

The quiver with the addition of these extra fields is shown in figure 13.

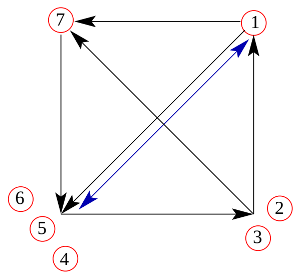

Figure 13: Quiver diagram for Model II of dP_4 showing in blue the fields that are missing from partial representations.

Now that we have identified the fields that are missing from the partial representations, we can rewrite the superpotential for this model in an expression that includes all the fields in the theory, both massless and massive. It becomes (note the mass term for the $\mathbf{5}$ and $\bar{\mathbf{5}}_\mathbf{b}$ representations)

$$\begin{aligned} W_{II} &= [10 \otimes \bar{5}_a \otimes \bar{5}_a + 10 \otimes 10 \otimes 5 + 10 \otimes \bar{5}_a \otimes \bar{5}_b] + \\ &\quad +10 \otimes \bar{5}_b \otimes \bar{5}_b + 5 \otimes \bar{5}_b \,. \end{aligned} \tag{4.2}$$

The products of representations between brackets are already present in (3.24). Keeping in mind that some of the fields in the $\mathbf{5}$ and $\bar{\mathbf{5}}_\mathbf{b}$ remain massless, it is straightforward to prove that the previous expression reduces to (3.24) (which only includes massless fields) when the massive fields are integrated out.

Let us now consider a different example. In Model II of dP_5, fields within a single representation are combined to form quadratic terms. The procedure described above can be applied without changes to this situation. The first step is to identify the Dynkin components and R charge of the missing fields. From them, the corresponding divisors are computed. Finally, the gauge charges of the missing fields are determined. This information is summarized in table (4.3).

$(J_1, J_2, J_3, J_4, J_5, R)$	Divisor	Bifundamental
$(-1, 0, 0, 0, 0, 1)$	$2D - E_2 - E_3 - E_4 - E_5$	X_{14}
$(1, -1, 0, 0, 0, 1)$	$2D - E_1 - E_3 - E_4 - E_5$	X_{15}
$(0, 1, -1, 0, 0, 1)$	$2D - E_1 - E_2 - E_4 - E_5$	X_{16}
$(1, 0, 0, 0, 0, 1)$	$D - E_1$	X_{41}
$(-1, 1, 0, 0, 0, 1)$	$D - E_2$	X_{51}
$(0, -1, 1, 0, 0, 1)$	$D - E_3$	X_{61}

(4.3)

Figure 14 shows where the fields that are missing from the partial $\mathbf{10}$ appear in the quiver for Model II of dP_5.

Including the massive fields, we can write the superpotential as

$$W_{II} = 16^2 \otimes 10 + 10^2 \tag{4.4}$$

which reproduces (3.30) when integrating out those fields in the **10** representation that are massive.

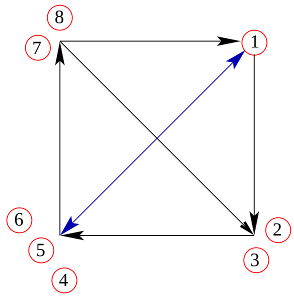

Figure 14: Quiver diagram for Model II of dP_5 showing in blue the fields that are missing from partial representations.

4.2 Partial representations and E_n subgroups

It is possible to make a further characterization of theories with partial representations using group theory. This is attained by considering the transformation properties of fields, both present and missing from the quiver, under subgroups of E_n. These subgroups appear when some of the nodes in a quiver theory fall into blocks. For a block containing n_i nodes, a subgroup $\mathrm{SU}(n_i)$ of the enhanced global symmetry E_n becomes manifest. In the general case, the manifest subgroup of the enhanced global symmetry will be a product of such $\mathrm{SU}(n_i)$ factors.[6] A matter field is charged under one of these factors if it is attached to one of the nodes in the corresponding block. Let us see how this works in the quivers where partial representations appear.

The first example is Model II of dP_4. Its quiver, shown in figure 8, features two blocks, with two and three nodes each. The corresponding subgroup of E_4 is $\mathrm{SU}(2) \times \mathrm{SU}(3)$. There are two partial representations, a **5** and a $\bar{\mathbf{5}}$. The way they decompose under this subgroup is

$$\begin{aligned} 5 &\to (2,1) + (1,3) \\ \bar{5} &\to (2,1) + (1,\bar{3})\,. \end{aligned} \tag{4.5}$$

The fields which do not appear in the low energy limit are the ones in the $(1,3)$ and $(1,\bar{3})$ and run from nodes 4, 5, 6 to node 1 and vice versa, as was derived in section 4.1. These fields form a quadratic gauge invariant which is a mass term and is the singlet in the product $(1,3) \otimes (1,\bar{3})$. The singlet in $(2,1) \otimes (2,1)$ is not gauge invariant so these fields are massless and make up the partial representations. The important observation is that fields present and missing from the quiver can be organized in representations of these relatively small subgroups of E_n, their classification becomes simpler.

A similar situation occurs for Model II of dP_5, depicted in figure 10. The subgroup here is $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ and we have a partial **10** representation which decomposes as

$$10 \to (2,2,1) + (1,1,3) + (1,1,\bar{3})\,. \tag{4.6}$$

The missing fields are the ones in $(1,1,3)$ and $(1,1,\bar{3})$, extending from node 1 to nodes $4, 5, 6$ and vice-versa respectively in the quiver. As before, the singlet of $(1,1,3) \otimes (1,1,\bar{3})$ is a quadratic gauge invariant that makes these fields massive.

[6] dP_n quivers have $n+3$ nodes. For each block of n_i nodes, the associated $\mathrm{SU}(n_i)$ factor has rank $n_i - 1$. Thus, we see that for the specific case of 3-block quivers the sum of the ranks of the three $\mathrm{SU}(n_i)$ factors is n, corresponding to a maximal subgroup of the corresponding E_n.

The last example where partial representations occur is Model III of dP_5. This is a three block model and the manifest subgroup of E_5 is SU(2) × SU(2) × SU(4), which is maximal. The **10** of SO(10) breaks as

$$10 \rightarrow (2,2,1) + (1,1,6)\,. \tag{4.7}$$

The four fields that appear in each of the two partial **10**'s are the ones in $(2,2,1)$, running from nodes 5, 6 to nodes 7, 8. The matter fields in $(1,1,6)$ are charged only under the SU(4) factor and thus cannot extend between two blocks of nodes. These fields run between pairs of nodes in the SU(4) block (there are exactly six distinct pairs). The fields coming from the two partial representations run in opposite directions making the arrows between the nodes bidirectional. The corresponding superpotential terms are given by the singlet of $(1,1,6) \otimes (1,1,6)$ and make the fields massive.

5. Higgsing

It is interesting to understand how the gauge theories for different del Pezzos are related to each other. The transition from dP_n to dP_{n-1} involves the blow-down of a 2-cycle. This operation appears in the gauge theory as a higgsing, by turning a non-zero VEV for a suitable bifundamental field. The determination of possible choices of this field has been worked out case by case in the literature.

The (p,q) web techniques introduced in [32] provide us with a systematic approach to the higgsing problem. In these diagrams, finite segments represent compact 2-cycles. The blow down of a 2-cycle is represented by shrinking a segment in the web and the subsequent combination of the external legs attached to it. The bifundamental field that acquires a non-zero VEV corresponds to the one charged under the gauge groups associated to the legs that are combined. The reader is referred to [32] to a detailed explanation of the construction, interpretation and applications of (p,q) webs in the context of four dimensional gauge theories on D3-branes on singularities. (p,q) webs are traditionally associated to toric geometries, since they represent the reciprocal lattice of a toric diagram (see [33] for a recent investigation of the precise relation). Nevertheless, their range of applicability can be extended to the determination of quivers [31] and higgsings of non-toric del Pezzos.

In this section, we will derive all possible higgsings from dP_n down to dP_{n-1} using (p,q) webs. The passage from dP_3 to dP_2 will be discussed in detail, and the presentation of the results for other del Pezzos will be more schematic. After studying these examples, it will be clear that this determination becomes trivial when using the information about global symmetries summarized in section 3. The problem can be rephrased as looking for how to higgs the global symmetry group from E_n to E_{n-1} by giving a non-zero VEV to a field that transform as a non-singlet under E_n. We will conclude this section by writing down the simple group theoretic explanation of our findings.

5.1 Del Pezzo 3

As our first example, we proceed now to determine all possible higgsings from the four phases of dP_3 down to the two phases of dP_2.

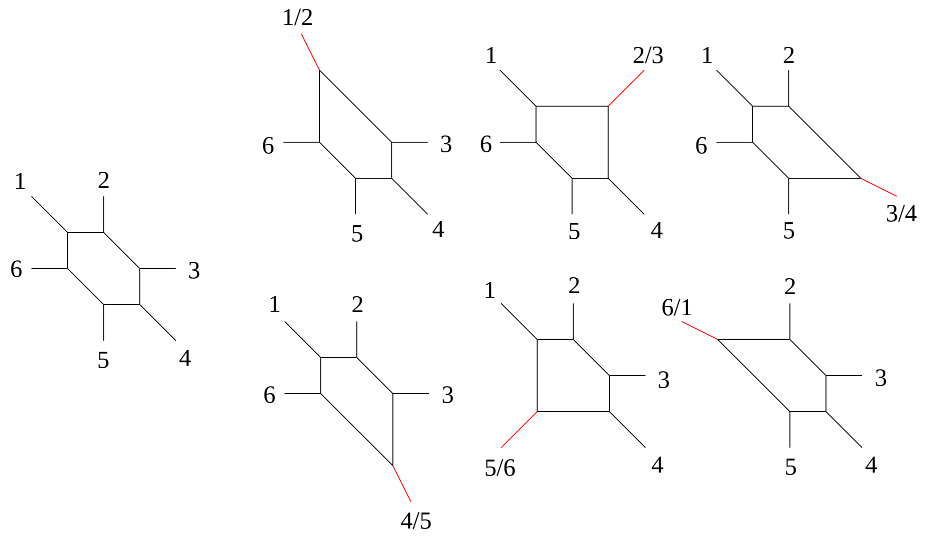

Figure 15: (p, q) webs describing all possible higgsings from Model I of dP_3 down to dP_2.

Model I. The (p, q) webs representing the higgsing of this phase down to dP_2 are presented in figure 15. We have indicated in red, the combined external legs that result from blowing down 2-cycles. All the resulting webs in figure 15 are related by $\mathrm{SL}(2, \mathbb{Z})$ transformations, implying that in this case it is only possible to obtain Model II of dP_2 by higgsing.

From the external legs that have to be combined in the (p, q) webs in figure 15, we conclude that the lowest component of the following bifundamental chiral superfields should get a non-zero VEV in order to produce the higgsing

$$X_{12}, X_{23}, X_{34}, X_{45}, X_{56}, X_{61} \rightarrow \text{Model II}\} \rightarrow (2,3)\,. \tag{5.1}$$

We notice that these fields form precisely a $(2, 3)$, i.e. a fundamental representation, of $E_3 = \mathrm{SU}(2) \times \mathrm{SU}(3)$. We will see that the same happens for all the del Pezzos.

Model II. The (p, q) web diagrams for this model are shown in figure 16, along with the possible higgsings.

It is important to remind the reader of the node symmetries that appear in the gauge theory when parallel external legs are present [32]. This situation appears in this example, and we have drawn only one representative of each family of (p, q) webs related by this kind of symmetries. In the language of exceptional collections, each set of parallel external legs corresponds to a block of sheaves. The full list of bifundamental fields associated to the higgsings in figure 16 is summarized in the following list

$$\left.\begin{array}{ll} X_{21}, X_{65} & \rightarrow \text{Model I} \\ X_{32}, X_{42}, X_{53}, X_{54} & \rightarrow \text{Model II} \end{array}\right\} \rightarrow (2,3)\,. \tag{5.2}$$

Model III. Figure 17 shows the (p, q) webs describing the higgsings of this phase. Once again, we have included only one representative of each set of webs related by node symmetries.

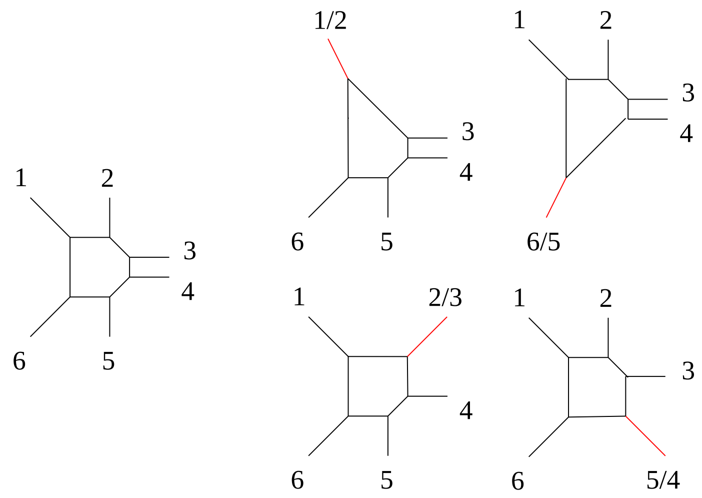

Figure 16: (p,q) webs describing all possible higgsings from Model II of dP_3 down to dP_2.

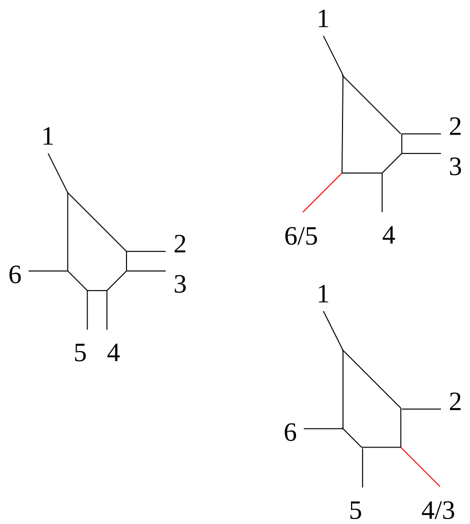

Figure 17: (p,q) webs describing all possible higgsings from Model III of dP_3 down to dP_2.

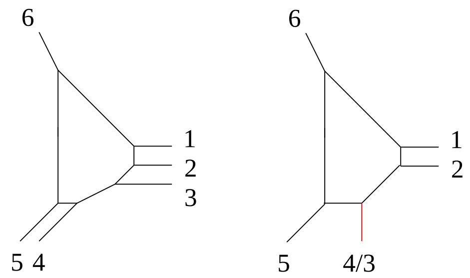

Figure 18: (p,q) webs describing all possible higgsings from Model IV of dP_3 down to dP_2.

Then, we have the following set of fields that produce a higgsing to dP_2.

$$\left.\begin{array}{ll} X_{64}, X_{65} & \rightarrow \text{Model I} \\ X_{42}, X_{43}, X_{52}, X_{53} & \rightarrow \text{Model II} \end{array}\right\} \rightarrow (2,3)\,. \tag{5.3}$$

Model IV. Finally, for Model IV of dP_3, we have the webs shown in figure 18 Indicating that the following fields can take us from this phase to dP_2.

$$X_{41}, X_{42}, X_{43}, X_{51}, X_{52}, X_{53} \rightarrow \text{Model I}\} \rightarrow (2,3)\,. \tag{5.4}$$

Having studied the four toric phases of dP_3, we see that for all of them the fields that produce a higgsing down to dP_2 are those in the fundamental $(2,3)$ representation of

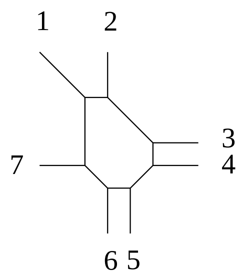

Figure 19: (p,q) web for Model I of dP_4.

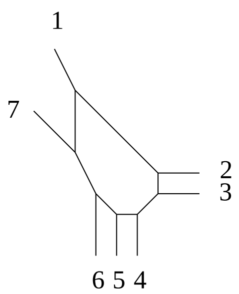

Figure 20: (p,q) web for Model II of dP_4.

$E_3 = \mathrm{SU}(2) \times \mathrm{SU}(3)$. We will show below how the higgsing of all other del Pezzos is also attained by a non-zero VEV of any field in the fundamental representation of E_n.

5.2 Del Pezzo 4

Let us now analyze the two toric phases of dP_4.

Model I. A possible (p,q) web for this model is the one in figure 19. From now on, for simplicity, we will only present the original webs but not the higgsed ones. From figure 19, we determine the following higgsings

$$\left.\begin{array}{ll} X_{35}, X_{36}, X_{45}, X_{46} & \rightarrow \text{Model I} \\ X_{23}, X_{24}, X_{57}, X_{67} & \rightarrow \text{Model II} \\ X_{12}, X_{71} & \rightarrow \text{Model III} \end{array}\right\} \rightarrow 10\,. \tag{5.5}$$

As expected, the fields that higgs the model down to some dP_3 phase form the fundamental **10** representation of E_4.

Model II. The (p,q) web for Model II is presented in figure 20.

The higgsings in this case become

$$\left.\begin{array}{ll} X_{42}, X_{52}, X_{62}, X_{43}, X_{53}, X_{63} & \rightarrow \text{Model II} \\ X_{74}, X_{75}, X_{76} & \rightarrow \text{Model III} \\ X_{17} & \rightarrow \text{Model IV} \end{array}\right\} \rightarrow 10\,. \tag{5.6}$$

5.3 Del Pezzo 5

There are three toric phases for dP_5. We study now their higgsing down to dP_4.

Model I. The web diagram corresponding to this theory is shown in figure 21. From it, we see that the fields that can take us to dP_4 by getting a non-zero VEV are

$$\left.\begin{array}{l} X_{13}, X_{14}, X_{23}, X_{24}, X_{35}, X_{36}, X_{45}, X_{46} \\ X_{57}, X_{58}, X_{67}, X_{68}, X_{71}, X_{72}, X_{81}, X_{82} \end{array} \rightarrow \text{Model I}\right\} \rightarrow 16\,. \tag{5.7}$$

Model II. The web in this case is shown in figure 22.

The fields that higgs the theory down to dP_4 are

$$\left.\begin{array}{ll} X_{24}, X_{25}, X_{26}, X_{34}, X_{35}, X_{36}, X_{47}, X_{57}, X_{67}, X_{48}, X_{58}, X_{68} & \rightarrow \text{Model I} \\ X_{12}, X_{13}, X_{71}, X_{81} & \rightarrow \text{Model II} \end{array}\right\} \rightarrow 16\,. \tag{5.8}$$

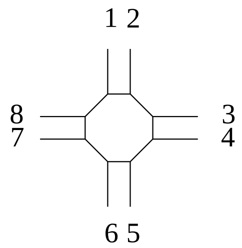

Figure 21: (p, q) web for Model I of dP_5.

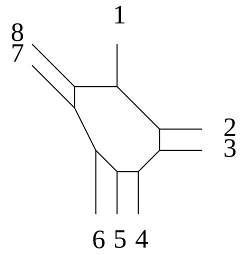

Figure 22: (p, q) web for Model II of dP_5.

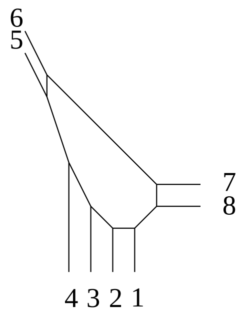

Figure 23: (p, q) web for Model III of dP_5.

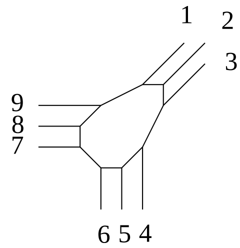

Figure 24: (p, q) web for dP_6.

Model III. From the (p, q) web in figure 23, we see that the following fields higgs the theory down to dP_4

$$\left.\begin{array}{l} X_{71}, X_{72}, X_{73}, X_{74}, X_{81}, X_{82}, X_{83}, X_{84} \\ X_{15}, X_{25}, X_{35}, X_{45}, X_{16}, X_{26}, X_{36}, X_{46} \end{array} \rightarrow \text{Model II} \right\} \rightarrow 16\,. \tag{5.9}$$

5.4 Del Pezzo 6

Finally, we present the web for dP_6 in figure 24, from where we read the following higgsings

$$\left.\begin{array}{l} X_{14}, X_{15}, X_{16}, X_{24}, X_{25}, X_{26}, X_{34}, X_{35}, X_{36} \\ X_{47}, X_{48}, X_{49}, X_{57}, X_{58}, X_{59}, X_{67}, X_{68}, X_{59} \\ X_{71}, X_{72}, X_{73}, X_{81}, X_{82}, X_{83}, X_{91}, X_{92}, X_{93} \end{array} \rightarrow \text{Model II} \right\} \rightarrow 27\,. \tag{5.10}$$

5.5 Higgsing global symmetry groups

The results above show that, once the we classify quiver theories using their global symmetry groups, the identification of higgsings that correspond to blow-downs of the geometry becomes straightforward. In particular, we have seen for each dP_n quiver theories that turning on a non-zero VEV for any component field of the fundamental representation reduces the global symmetry from E_n to E_{n-1} and produces a toric phase of dP_{n-1}.

We will now present a comprehensive discussion on the group theory considerations that lead to the choice of the appropriate bifundamental fields that acquire a non-zero VEV. We will also present a description of how the blow-down process corresponds to a Higgs mechanism for the relevant gauge groups, both from four and five dimensional perspectives.

In the previous section, we established which fields produce the desired higgsing with the aid of (p, q) webs, and discovered that in all cases they form the fundamental representation of the corresponding E_n group. In fact, it is possible to determine which representation to choose for higgsing using solely group theoretic considerations. Generically, more than one representation are present in a given quiver theory. The representations that appear are always basic, meaning that their highest weight vectors are simple roots of the algebra. The clue to the right representation is provided by the Dynkin diagrams for the E_n Lie algebras. The higgsing, and the corresponding enhanced symmetry breaking $E_n \to E_{n-1}$ can be depicted as the removal of a certain node from the E_n Dynkin diagram. The representation that must be used for the higgsing is the one corresponding to the removed node, in the sense that its highest weight vector is equal to the root corresponding to this node. Practically this means that the higgsed representation must contain a matter field with only one non-zero U(1) charge in the same position as the removed node in the Dynkin diagram.[7] The numbering of the nodes in the Dynkin diagram is unique in the basis of U(1)'s that we have chosen, leaving no space for ambiguity.

Let us now discuss the blow-down as a Higgs mechanism in four dimensions. As explained in previous sections, we go down from dP_n to dP_{n-1} by blowing down one of the exceptional divisors E_i $(i = 1 \dots n-1)$ of the del Pezzo. In fact, by acting on the E_i's with the Weyl group of E_n, other possible choices of cycles to blow down are generated. As we have seen, all these divisors form the fundamental representation of E_n and give precisely the divisors $L_{\alpha\beta}$ of the bifundamentals $X_{\alpha\beta}$ that can be used to appropriately higgs the gauge theory to one for dP_{n-1}. The bifundamental fields $X_{\alpha\beta}$ transform in the $(\bar{N}, N)$ representation of the four dimensional gauge groups $\mathrm{U}(N)_{(\alpha)} \times \mathrm{U}(N)_{(\beta)}$ of the nodes they connect. Blowing down $L_{\alpha\beta}$ corresponds to $X_{\alpha\beta}$ getting a VEV proportional to the $N \times N$ identity matrix, higgsing $\mathrm{U}(N)_{(\alpha)} \times \mathrm{U}(N)_{(\beta)}$ to the diagonal subgroup. The non-zero VEV introduces a scale in the otherwise conformal field theory, and the new quiver will correspond to a new fixed point at the IR limit of the renormalization group flow.

On the other hand, bifundamental fields are also charged under the baryonic U(1) global symmetries, which correspond to gauge symmetries in AdS_5. Then, the blow-down can also be interpreted as Higgs mechanism of a different gauge group, this time in five dimensions. The original five dimensional gauge group is in this case $\mathrm{U}(1)^n$ and is higgsed down to $\mathrm{U}(1)^{(n-1)}$. The simplest case corresponds to $X_{\alpha\beta}$ having charge q_1 under a factor $\mathrm{U}(1)_{(1)}$ and q_2 under $\mathrm{U}(1)_{(2)}$. Then, the VEV of $X_{\alpha\beta}$ higgses $\mathrm{U}(1)_{(1)} \times \mathrm{U}(1)_{(2)}$ to $q_2\, \mathrm{U}(1)_{(1)} - q_1\, \mathrm{U}(1)_{(2)}$, while the orthogonal combination becomes massive. The general case, in which the bifundamental field is charged under more than two U(1)'s is analogous, and simply amounts to a different combination of the original $\mathrm{U}(1)'s$ becoming massive.

Let us conclude this section with a few words describing how the enhanced E_n global symmetry group of dP_n is related to the E_{n-1} group of dP_{n-1}. BPS states correspond to certain limits of D3-branes wrapping 3-cycles with $S^2 \times S^1$ topology in the non-spherical horizon H_5. In particular, these wrapped branes give rise to the charged gauge bosons

[7] The case $E_3 \to E_2$ is slightly special, in that we have to remove two nodes from the (disconnected) Dynkin diagram. An obvious generalization of the described procedure shows that the representation we have to higgs is the (2,3).

in AdS_5 that, together with the generators in the Cartan subalgebra that come from the reduction over 3-cycles of the RR C_4, generate an E_n gauge symmetry on AdS_5. The masses of the five dimensional W bosons, as well as the four dimensional gauge couplings, are Kähler moduli. The E_n enhancement only appears at infinite gauge coupling. Finite couplings produce an adjoint higgsing of E_n down to $\mathrm{U}(1)^n$, giving mass to the W bosons. We have discussed above how $\mathrm{U}(1)^n$ is connected to $\mathrm{U}(1)^{n-1}$ by higgsing with a bifundamental field in the quiver. Then, we see that the global symmetry groups of dP_n and dP_{n-1} at finite and infinite coupling are connected as follows

$$\begin{array}{ccccc} & & \begin{array}{c}\text{bifundamental}\\ \text{higgsing}\end{array} & & \\ & E_n & \longrightarrow & E_{n-1} & \\ \begin{array}{c}\text{finite coupling}\\ \text{(adjoint higgsing)}\end{array} & \downarrow & & \downarrow & \begin{array}{c}\text{finite coupling}\\ \text{(adjoint higgsing)}\end{array} \\ & \mathrm{U}(1)^n & \longrightarrow & \mathrm{U}(1)^{n-1} & \\ & & \begin{array}{c}\text{bifundamental}\\ \text{higgsing}\end{array} & & \end{array} \tag{5.11}$$

where we have included the possibility of connecting the exceptional groups at infinite coupling directly.

6. Global symmetries and Seiberg duality

Section 3 explained how to determine the divisors associated to bifundamental fields when blowing-up or blowing-down 2-cycles or when performing a Seiberg duality. We also saw there how the intersections with the divisors generating the U(1) flavor symmetries determine the E_n Dynkin labels for each bifundamental field. Therefore, given the representation structure of an original gauge theory, it is possible to determine how a Seiberg dual quiver is organized into E_n representations, by carefully following this algorithm.

The purpose of this section is to give a straightforward alternative procedure to determine the transformation properties under global symmetries of fields in a Seiberg dual theory. It consists of three simple rules, which have their origin in how E_n symmetries are realized in the quivers, and also admits a geometric interpretation. Before going on, it is important to remind the reader the key fact that the bifundamental fields transforming in an irreducible representation of the E_n global symmetry group do not necessarily have the same gauge quantum numbers (i.e. they can be charged under different pairs of gauge groups).

The three steps to deduce the representation structure of a theory based on that of a Seiberg dual are:

- *Step 1:* fields that are neutral under the dualized gauge group remain in the same representations. Some places in those representations might be left empty by the fields (otherwise known as dual quarks) that are conjugated (their transformation properties are yet to be determined) and by fields that become massive. If the representation is such that it cannot appear in partial form (i.e. it is not possible to combine it with other representation or with itself to form a quadratic invariant), these places will be completed either by meson fields or the conjugated ones (dual quarks).

- *Step 2:* Seiberg mesons appear in the product of the representations of the constituent fields. The precise representation is chosen from all the ones appearing in the product by requiring that those superpotential terms that include the mesons are singlets under the global symmetry group. The geometric interpretation of this step is that the divisor associated to mesons is given by the composition of the two divisors corresponding to the component fields. As we studied above, the requirement that the superpotential terms are E_n singlets is translated to the geometric condition that the associated divisor lays in the canonical class.

- *Step 3:* the representations for the conjugated fields are determined by requiring that the cubic meson terms added to the superpotential are singlets of the global symmetry group. When doing so, it is very useful to choose these representations based on the entries that were left vacant at step **(1)**, if this is possible. Once again, the geometric perspective is that the sum of the divisors appearing in a superpotential term should be in the canonical class.

We will now use this technique to determine the global symmetries for all toric phases starting from one of them, for del Pezzo surfaces from dP_2 to dP_5. As we will see, the results obtained this way are consistent with the Dynkin components assignations listed in section 3, obtained using the geometric prescription.

6.1 Del Pezzo 2

6.1.1 Model II of dP_2

Let us start from Model I and dualize on node 1. In this case, we will have

$$\begin{array}{cc} X_{41} \to X_{14} & X_{51} \to X_{15} \\ X_{13} \to X_{31} & Y_{13} \to Y_{31} \end{array} \tag{6.1}$$

and the following Seiberg mesons have to be added

$$\begin{array}{ll} M_{43} = X_{41}X_{13} & \tilde{M}_{43} = X_{41}Y_{13} \\ M_{53} = X_{51}X_{13} & \tilde{M}_{53} = X_{51}Y_{13} \end{array} \tag{6.2}$$

When dualizing, the fields which become massive, and have to be integrated out using their equations of motion are

$$X_{34}\,, \quad M_{43}\,, \quad Y_{35}\,, \quad Z_{35}\,, \quad M_{53}\,, \quad \tilde{M}_{53}\,. \tag{6.3}$$

Next we demonstrate the three steps.

JHEP07(2004)060

Step 1: fields that are neutral under the dualized gauge group remain invariant.

$$\begin{array}{cc} & \mathrm{SU}(2)\times\mathrm{U}(1) \\ (*, X_{42}) & 2_3 \\ (X_{52}, *) & 2_{-1} \\ (*, X_{23}) & 2_{-1} \\ (*, Y_{23}) & 2_{-1} \\ X_{35} & 1_2 \\ X_{54} & 1_{-4} \end{array} \tag{6.4}$$

where we have left an empty space for each conjugated or massive field.

Step 2: Seiberg mesons are composite fields, and thus transform in an irreducible representation in the product of their constituents. The right representation is chosen from the existent terms in the superpotential

$$\begin{aligned} M_{43} &= X_{41}X_{13} = 2_3 \otimes 2_{-1} = (1 \oplus 3)_2 \to 1_2 \\ \tilde{M}_{43} &= X_{41}Y_{13} = 2_3 \otimes 2_{-1} = (1 \oplus 3)_2 \to 1_2 \\ M_{53} &= X_{51}X_{13} = 2_{-1} \otimes 2_{-1} = (1 \oplus 3)_{-2} \to 1_{-2} \\ \tilde{M}_{53} &= X_{51}Y_{13} = 2_{-1} \otimes 2_{-1} = (1 \oplus 3)_{-2} \to 1_{-2} \end{aligned} \tag{6.5}$$

Step 3: we still have to determine the transformation properties of X_{14}, X_{15}, X_{31} and Y_{31}. This is done by requiring that the meson terms added to the superpotential are invariant under the $\mathrm{SU}(2) \times \mathrm{U}(1)$ transformations

$$M_{43}X_{31}X_{14} + \tilde{M}_{43}Y_{31}X_{14} + M_{53}X_{31}X_{15} + \tilde{M}_{53}Y_{31}X_{15} \tag{6.6}$$

from were we conclude that the fields transform according to

$$\begin{array}{ll} X_{14} \in 2_{-1} & X_{15} \in 2_3 \\ X_{31} \in 2_{-1} & Y_{31} \in 2_{-1} \end{array} \tag{6.7}$$

resulting in the following arrangement of the bifundamental chiral fields

$$\begin{array}{cc} & \mathrm{SU}(2)\times\mathrm{U}(1) \\ (X_{15}, X_{42}) & 2_3 \\ (X_{52}, X_{14}) & 2_{-1} \\ (X_{31}, X_{23}) & 2_{-1} \\ (Y_{31}, Y_{23}) & 2_{-1} \\ X_{35} & 1_2 \\ \tilde{M}_{43} & 1_2 \\ X_{54} & 1_{-4} \end{array} \tag{6.8}$$

After the renaming of nodes indicated by the quiver diagram $(1,2,3,4,5)\to(2,5,1,4,3)$ we see that we recover the results in (3.5), derived using geometric methods.

6.2 Del Pezzo 3

Let us repeat the program we used for dP_2 in the case of dP_3, deriving the symmetry properties of Models II, III and IV from Model I.

6.2.1 Model II of dP_3

Model II is obtained from Model I by dualizing node 1. We get

$$\begin{array}{ll} X_{12} \to X_{21} & X_{13} \to X_{31} \\ X_{51} \to X_{15} & X_{61} \to X_{16} \end{array} \tag{6.9}$$

and the following mesons are added

$$\begin{array}{ll} M_{52} = X_{51}X_{12} & M_{53} = X_{51}X_{13} \\ M_{62} = X_{61}X_{12} & M_{63} = X_{61}X_{13} \end{array} \tag{6.10}$$

Proceeding as before

Step 1:

$$\begin{array}{cc} & \mathrm{SU}(2) \times \mathrm{SU}(3) \\ (*, X_{23}, X_{34}, X_{45}, X_{56}, *) & (2,3) \\ (X_{24}, X_{46}, X_{62}) & (1,\bar{3})\,. \end{array} \tag{6.11}$$

One of the $(\mathbf{1}, \bar{\mathbf{3}})$ representations has completely disappeared, while the other one stays unchanged. Note that M_{53} and X_{35} become massive and are in complex conjugate representations.

Step 2:

$$\begin{aligned} M_{52} &= X_{51}X_{12} = (1,\bar{3}) \otimes (2,3) = (2, 1 \oplus 8) \to (2,1) \\ M_{53} &= X_{51}X_{13} = (1,\bar{3}) \otimes (1,\bar{3}) = (1, 3 \oplus \bar{6}) \to (1,3) \\ M_{62} &= X_{61}X_{12} = (2,3) \otimes (2,3) = (1 \oplus 3, \bar{3} \oplus 6) \to (1,\bar{3}) \\ M_{63} &= X_{61}X_{13} = (2,3) \otimes (1,\bar{3}) = (2, 1 \oplus 8) \to (2,1) \end{aligned} \tag{6.12}$$

Step 3: in order to determine the transformation properties of X_{21}, X_{31}, X_{15} and X_{16}, we study the meson terms in the superpotential

$$M_{52}X_{21}X_{15} + M_{53}X_{31}X_{15} + M_{62}X_{21}X_{16} + M_{63}X_{31}X_{16}\,. \tag{6.13}$$

Thus, we conclude that

$$\begin{array}{ll} X_{21} \in (1,\bar{3}) & X_{31} \in (2,3) \\ X_{15} \in (2,3) & X_{16} \in (1,\bar{3})\,. \end{array} \tag{6.14}$$

Putting all these results together

$$\begin{array}{cc} & \mathrm{SU}(2) \times \mathrm{SU}(3) \\ (X_{31}, X_{23}, X_{34}, X_{45}, X_{56}, X_{15}) & (2,3) \\ (X_{24}, X_{46}, X_{62}) & (1,\bar{3}) \\ (M_{62}, X_{21}, X_{16}) & (1,\bar{3}) \\ (M_{52}, M_{63}) & (2,1) \end{array} \tag{6.15}$$

which becomes (3.11) after relabeling the gauge groups according to $(1,2,3,4,5,6) \to (4,6,5,3,2,1)$.

6.2.2 Model III of dP_3

Model III is obtained by dualizing node 5 of Model II. The dual quarks are

$$\begin{aligned} X_{15} &\to X_{51} & X_{65} &\to X_{56} \\ X_{54} &\to X_{45} & X_{53} &\to X_{35}\,. \end{aligned} \tag{6.16}$$

The Seiberg mesons are

$$\begin{aligned} M_{13} &= X_{15}X_{53} & M_{14} &= X_{15}X_{54} \\ M_{63} &= X_{65}X_{53} & M_{64} &= X_{65}X_{54}\,. \end{aligned} \tag{6.17}$$

The following fields become massive and are integrated out: X_{31}, M_{13}, X_{41} and M_{14}. Let us now apply our set of rules.

Step 1:

$$\begin{array}{cc} & \mathrm{SU}(2)\times\mathrm{SU}(3) \\ (*,*,*,X_{32},X_{21},X_{42}) & (2,3) \\ (X_{63},*,X_{16}) & (1,\bar{3}) \\ (X_{64},*,Y_{16}) & (1,\bar{3}) \\ (X_{26},*) & (2,1) \end{array} \tag{6.18}$$

Step 2:

$$\begin{aligned} M_{13} &= X_{15}X_{53} = (2,1)\otimes(2,3) = (1\oplus 3,3) \to (1,3) \\ M_{14} &= X_{15}X_{54} = (2,1)\otimes(2,3) = (1\oplus 3,3) \to (1,3) \\ M_{63} &= X_{65}X_{53} = (2,3)\otimes(2,3) = (1\oplus 3,\bar{3}\oplus 6) \to (1,\bar{3}) \\ M_{64} &= X_{65}X_{54} = (2,3)\otimes(2,3) = (1\oplus 3,\bar{3}\oplus 6) \to (1,\bar{3}) \end{aligned} \tag{6.19}$$

Step 3: we now determine the representations for X_{51}, X_{56}, X_{35} and X_{45}. The new terms in the superpotential are

$$M_{13}X_{35}X_{51} + M_{14}X_{45}X_{51} + M_{63}X_{35}X_{56} + M_{64}X_{45}X_{56}\,. \tag{6.20}$$

Then,

$$\begin{aligned} X_{51} &\in (2,3) & X_{56} &\in (2,1) \\ X_{35} &\in (2,3) & X_{45} &\in (2,3)\,. \end{aligned} \tag{6.21}$$

Putting all the fields together we have

$$\begin{array}{cc} & \mathrm{SU}(2)\times\mathrm{SU}(3) \\ (X_{51},X_{35},X_{45},X_{32},X_{21},X_{42}) & (2,3) \\ (X_{63},M_{64},X_{16}) & (1,\bar{3}) \\ (M_{63},X_{64},Y_{16}) & (1,\bar{3}) \\ (X_{26},X_{56}) & (2,1) \end{array} \tag{6.22}$$

which becomes (3.14) after renaming the gauge groups according to $(1,2,3,4,5,6) \to (6,2,3,1,4,5)$.

6.2.3 Model IV of dP_3

Model IV can be obtained for example by dualizing node 6 of Model III. Then,

$$\begin{aligned} X_{16} &\to X_{61} \qquad & Y_{16} &\to Y_{61} \\ X_{64} &\to X_{46} \qquad & X_{65} &\to X_{56}\,. \end{aligned} \tag{6.23}$$

The following mesons are added

$$\begin{aligned} M_{14} &= X_{16}X_{64} \qquad & \tilde{M}_{14} &= Y_{16}X_{64} \\ M_{15} &= X_{16}X_{65} \qquad & \tilde{M}_{15} &= Y_{16}X_{65} \end{aligned} \tag{6.24}$$

There are no fields that become massive in this case.

Step 1:

$$\begin{array}{cc} & \mathrm{SU}(2)\times\mathrm{SU}(3) \\ (X_{42},*,X_{43},X_{53},*,X_{52}) & (2,3) \\ (X_{21},*,X_{31}) & (1,\bar{3}) \\ (Y_{21},*,Y_{31}) & (1,\bar{3}) \\ (X_{15},X_{14}) & (2,1) \end{array} \tag{6.25}$$

Step 2: Meson fields transform according to

$$\begin{aligned} M_{14} &= X_{16}X_{64} = (1,\bar{3})\otimes(2,3) = (2,1\oplus 8) \to (2,1) \\ \tilde{M}_{14} &= Y_{16}X_{64} = (1,\bar{3})\otimes(2,3) = (2,1\oplus 8) \to (2,1) \\ M_{15} &= X_{16}X_{65} = (1,\bar{3})\otimes(2,3) = (2,1\oplus 8) \to (2,1) \\ \tilde{M}_{15} &= Y_{16}X_{65} = (1,\bar{3})\otimes(2,3) = (2,1\oplus 8) \to (2,1)\,. \end{aligned} \tag{6.26}$$

Step 3: we determine the representations for X_{46}, X_{56}, X_{61} and Y_{61} by requiring the meson superpotential terms to be invariant

$$M_{14}X_{46}X_{61} + \tilde{M}_{14}X_{46}Y_{61} + M_{15}X_{56}X_{61} + \tilde{M}_{15}X_{56}Y_{61}\,. \tag{6.27}$$

And we see that

$$\begin{aligned} X_{46} &\in (2,3) \qquad & X_{56} &\in (2,3) \\ X_{61} &\in (1,\bar{3}) \qquad & Y_{61} &\in (1,\bar{3}) \end{aligned} \tag{6.28}$$

leading to

$$\begin{array}{cc} & \mathrm{SU}(2)\times\mathrm{SU}(3) \\ (X_{42},X_{46},X_{43},X_{53},X_{56},X_{52}) & (2,3) \\ (X_{21},X_{61},X_{31}) & (1,\bar{3}) \\ (Y_{21},Y_{61},Y_{31}) & (1,\bar{3}) \\ (X_{15},X_{14}) & (2,1) \\ (M_{15},M_{14}) & (2,1) \\ (\tilde{M}_{15},\tilde{M}_{14}) & (2,1) \end{array} \tag{6.29}$$

that reduces to (3.17) by renaming $(1,2,3,4,5,6)\to(6,1,3,5,4,2)$.

JHEP07(2004)060

6.3 Del Pezzo 4

We will now derive the global symmetry properties of Model II of dP_4 from those of Model I.

6.3.1 Model II of dP_4

The preceding examples show in detail how to operate with the rules in section 6 and classify the matter content of dual theories according to their global symmetry properties. We will now move on and apply our program to Model II of dP_4. This example is of particular interest because, as we mentioned in sections 3 and 4, it is the first one to exhibit *partial representations*.

We obtain Model II by dualizing Model I on node 7. As usual, those bifundamental fields that are charged under the dualized gauge group reverse their orientation

$$\begin{aligned} X_{71} &\to X_{17} \qquad & X_{72} &\to X_{27} \\ X_{57} &\to X_{75} \qquad & X_{67} &\to X_{76}\,. \end{aligned} \tag{6.30}$$

The following meson fields have to be incorporated

$$\begin{aligned} M_{51} &= X_{57}X_{71} \qquad & M_{52} &= X_{57}X_{72} \\ M_{61} &= X_{67}X_{71} \qquad & M_{62} &= X_{67}X_{72}\,. \end{aligned} \tag{6.31}$$

There are no fields in this theory that become massive, so we end up in a theory with 19 fields. At this point, we see the first indications that this model is rather peculiar since, as long as singlets are not used, it seems impossible to arrange these 19 fields into a combination of SU(5) representations. Let us apply the three step program as before.

Step 1:

$$\begin{array}{cc} \text{Fields} & \text{SU(5)} \\ (X_{45}, X_{23}, X_{46}, *, X_{36}, X_{24}, *, X_{35}, X_{12}, *) & 10 \\ (X_{51}, *, X_{61}, X_{13}, X_{14}) & \bar{5} \end{array} \tag{6.32}$$

Step 2:

$$\begin{aligned} M_{51} &= X_{57}X_{71} = 10 \otimes 10 = \bar{5} \oplus \bar{45} \oplus \bar{50} \to \bar{5} \\ M_{52} &= X_{57}X_{72} = 10 \otimes \bar{5} = 5 \oplus 45 \to 5 \\ M_{61} &= X_{67}X_{71} = 10 \otimes 10 = \bar{5} \oplus \bar{45} \oplus \bar{50} \to \bar{5} \\ M_{62} &= X_{67}X_{72} = 10 \otimes \bar{5} = 5 \oplus 45 \to 5\,. \end{aligned} \tag{6.33}$$

Both the $\mathbf{5}$ and $\bar{\mathbf{5}}$ representations will be partially filled with two fields each. The same number of fields are missing in both representations, as explained in section 4.

Step 3: looking at the superpotential terms

$$M_{51}X_{17}X_{75} + M_{52}X_{27}X_{75} + M_{61}X_{17}X_{76} + M_{62}X_{27}X_{76} \tag{6.34}$$

we see that

$$\begin{aligned} X_{27} &\in 10 \qquad & X_{17} &\in \bar{5} \\ X_{75} &\in \bar{5} \qquad & X_{76} &\in 10 \end{aligned} \tag{6.35}$$

and we obtain (3.23). We thus see that partial representations appear naturally when we study the transformation of theories under Seiberg duality.

6.4 Del Pezzo 5

We will obtain in this section the global symmetry structure of Models II and III of dP_5 from Model I.

6.4.1 Model II of dP_5

Model II is obtained by dualizing Model I on node 2.

$$\begin{aligned} X_{23} &\to X_{32} \qquad & X_{24} &\to X_{42} \\ X_{72} &\to X_{27} \qquad & X_{82} &\to X_{28}\,. \end{aligned} \tag{6.36}$$

The following mesons appear

$$\begin{aligned} M_{73} &= X_{72}X_{23} \qquad & M_{74} &= X_{72}X_{24} \\ M_{83} &= X_{82}X_{23} \qquad & M_{84} &= X_{82}X_{84}\,. \end{aligned} \tag{6.37}$$

No fields become massive.

Step 1:

$$\begin{array}{cc} & \mathrm{SO}(10) \\ \begin{array}{c}(X_{81}, X_{57}, *, X_{36}, X_{46}, *, X_{58}, X_{13}, \\ X_{14}, X_{71}, X_{45}, X_{35}, *, *, X_{67}, X_{68})\end{array} & 16 \end{array} \tag{6.38}$$

Step 2:

$$\begin{aligned} M_{73} &= X_{72}X_{23} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \\ M_{74} &= X_{72}X_{24} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \\ M_{83} &= X_{82}X_{23} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \\ M_{84} &= X_{82}X_{24} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \end{aligned} \tag{6.39}$$

Step 3: the representations for X_{32}, X_{42}, X_{27} and X_{28} are determined by considering the superpotential terms

$$M_{73}X_{32}X_{27} + M_{74}X_{42}X_{27} + M_{83}X_{32}X_{28} + M_{84}X_{42}X_{28}\,. \tag{6.40}$$

Thus,

$$X_{32}, X_{42}, X_{27}, X_{28} \in 16 \tag{6.41}$$

and we conclude that the matter is arranged as in (3.29).

6.4.2 Model III of dP_5

Dualizing Model II on node 1 we get Model III.

$$\begin{array}{ll} X_{71} \to X_{17} & X_{81} \to X_{18} \\ X_{12} \to X_{21} & X_{13} \to X_{31} \,. \end{array} \tag{6.42}$$

The Seiberg mesons are

$$\begin{array}{ll} M_{72} = X_{71}X_{12} & M_{82} = X_{81}X_{12} \\ M_{73} = X_{71}X_{13} & M_{83} = X_{81}X_{13} \,. \end{array} \tag{6.43}$$

There are no massive fields.

Step 1:

$$\begin{array}{cc} & \mathrm{SO}(10) \\ \begin{array}{c}(*, X_{25}, X_{24}, X_{67}, X_{68}, X_{34}, X_{35}, *, \\ *, *, X_{58}, X_{57}, X_{48}, X_{47}, X_{26}, X_{36})\end{array} & 16 \\ (X_{73}, X_{83}, X_{72}, X_{82}, *, *, *, *, *, *) & \text{partial } 10 \end{array} \tag{6.44}$$

Step 2:

$$\begin{aligned} M_{72} &= X_{71}X_{12} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \\ M_{82} &= X_{81}X_{12} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \\ M_{73} &= X_{71}X_{13} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \\ M_{83} &= X_{81}X_{13} = 16 \otimes 16 = 10 \oplus 120 \oplus 126 \to 10 \,. \end{aligned} \tag{6.45}$$

Step 3: looking at the following superpotential terms, we determine the representations for X_{17}, X_{18}, X_{21} and X_{31}.

$$M_{72}X_{21}X_{17} + M_{82}X_{21}X_{18} + M_{73}X_{31}X_{17} + M_{83}X_{31}X_{18} \,. \tag{6.46}$$

Then,

$$X_{17}, X_{18}, X_{21}, X_{31} \in 16 \tag{6.47}$$

and we recover (3.32).

7. Dibaryon operators

Section 8 will be devoted to another application of our E_n classification of quivers, the counting of dibaryon operators. These operators were introduced in section 2, where we also discussed their AdS realization as D3-branes wrapping 3-cycles in H_5. Equation (2.6) presented a special type of dibaryons, formed by antisymmetrizing N copies of a single kind of bifundamental field. Quiver theories admit a larger variety of dibaryons, which can be constructed from more complicated paths or subquivers [19]. Therefore, generic quivers can have dibaryons that correspond to bifurcated paths. This situation complicates the direct application of our techniques. We will restrict ourselves to two types of quivers, in which computations are relatively simple.

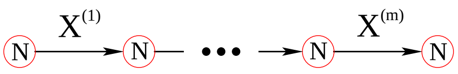

Figure 25: Path associated to a dibaryon in a toric quiver.

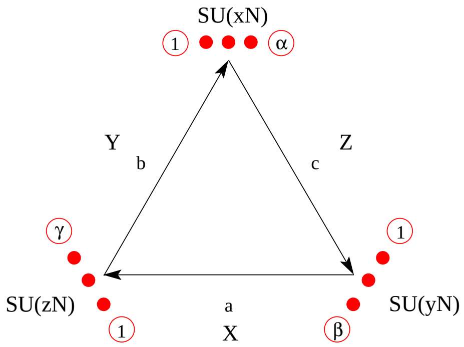

Figure 26: Quiver diagram for a generic 3-block collection.

The first class corresponds to the toric quivers we have studied so far. In these quivers, all gauge groups are identical. This prevents bifurcations in the paths that represent dibaryons, leaving us with linear paths as the one shown in figure 25

Dibaryons in these models take the form

$$\epsilon_{\alpha_1\cdots\alpha_N}\epsilon^{\beta_1\cdots\beta_N} X^{\alpha_1(1)}_{i_1}\cdots X^{\alpha_N(1)}_{i_N} X^{i_1(2)}_{j_1}\cdots X^{i_N(2)}_{j_N}\cdots X^{k_1(m)}_{\beta_1}\cdots X^{k_N(m)}_{\beta_N}\,. \tag{7.1}$$

There are exactly N copies of each $X^{(i)}$. In this sense, these dibaryons are analogous to the simple ones in (2.6) and, for the purposes of assigning divisors or representation under the Weyl group of E_n, we can consider just one representation for each of the arrows in figure 25.

The second class of theories that we will consider are 3-block quivers. Some of the examples that we will study are non-toric quivers. They have the general structure shown in figure 26, with no bifundamental fields connecting nodes in the same block.

These quivers are obtained as solutions of a Diophantine (Markov type) equation of the form [39]

$$\alpha x^2 + \beta y^2 + \gamma z^2 = \sqrt{K_n^2 \alpha\beta\gamma} xyz \tag{7.2}$$

where α, β and γ are the number of gauge groups in each of the three blocks, K_n denotes the canonical class as before and $K_n^2 = 9 - n$. The number of bifundamental fields between the nodes are computed as

$$a = \alpha k x \qquad b = \beta k y \qquad c = \gamma k z \tag{7.3}$$

with

$$k = \sqrt{\frac{K_n^2}{\alpha\beta\gamma}} = \sqrt{\frac{9-n}{\alpha\beta\gamma}}\,. \tag{7.4}$$

As a result, the arrow numbers satisfy the following Markov type equation.

$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = abc\,. \tag{7.5}$$

This equation is a reduction of the general Diophantine equation which was derived for a generic del Pezzo quiver in [36] to the case in which the corresponding (p, q) branes are grouped into 3 sets of parallel branes, of multiplicities α, β, γ, respectively. It is interesting to point out that equations (7.2) and (7.5) are derived by demanding that the quiver theory of figure 26 will be conformally invariant. Equation (7.3) has the interpretation of the anomaly cancellation condition for the quiver theories. The positive integers α, β and γ are solutions to

$$\alpha + \beta + \gamma = n + 3 , \qquad \alpha\beta\gamma(9 - n) = (\text{integer})^2 . \tag{7.6}$$

In addition these numbers represent all maximal subgroups of E_n, of A type,

$$\mathrm{SU}(\alpha) \times \mathrm{SU}(\beta) \times \mathrm{SU}(\gamma) \subset E_n , \tag{7.7}$$

with the convention that SU(1) is null. As an example, for E_8 there are 4 such triples: $(1, 1, 9)$, $(1, 2, 8)$, $(1, 5, 5)$, $(2, 3, 6)$.

We will study examples of non-toric 3-block quivers. An elegant discussion of the construction of dibaryons in general quivers can be found in [19]. There, a vector space V_i of dimension $d_i N$ is associated to each node, and every bifundamental field represents a linear map between its tail and its head. These maps can be multiplied to form linear maps A for arbitrary paths in the quiver associated to a given dibaryon, forming an algebra. Generically, these paths can have bifurcations. We show an example in figure 27.

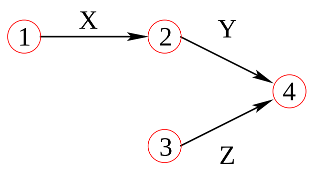

Figure 27: An example of a possible path in a quiver associated to a dibaryon. Nodes 2 and 3 can actually be the same.

A useful quantity that can be associated to a given path A is its rank

$$r(A) = \sum_i N d_i (h_i - t_i) , \tag{7.8}$$

where h_i and d_i are respectively the number of arrow heads and tails at each node. The rank of A counts the number of uncontracted fundamental minus antifundamental $\mathrm{SU}(d_i N)$ indices at each node. When $r(A) = 0$, the antisymmetrized product that gives rise to the dibaryon is simply given by the determinant of the linear map A between the vector spaces of the path's tail and head.

We have discussed how for non-toric quivers the baryonic U(1) charges are generically fractional, and cannot be interpreted as Dynkin coefficients. We will see in section 9 how in these cases it is still possible to exploit the decomposition of the global symmetry into subgroups to efficiently organize the counting of dibaryons.

The operators defined above, both for toric and general 3-block quivers do not have definite transformation properties under the global symmetry group. In order to achieve that, we have to construct linear combinations of them using appropriate Clebsch-Gordon coefficients. The techniques described in the next section will help us to directly organize dibaryons of different dimensions into representations of the Weyl groups of E_n, based on the classification of bifundamental fields.

8. Dibaryon counting

We have already seen how the classification into an E_n symmetric language has proved useful in simplifying and organizing some problems in dP_n quivers. It has reduced the computation of superpotentials to the construction of E_n invariants and the determination of possible higgsings from dP_n to dP_{n-1} to the problem of higgsing the global symmetry group by a non-zero VEV for a field in the fundamental representation. We will explore in this section an additional application of this machinery, using it to count dibaryon operators in the gauge theories. Matching the counting performed in the gauge theory with the one on the gravity side is another check of the AdS/CFT correspondence.

8.1 Geometric counting

The *geometric counting* of dibaryons corresponds to the enumeration of holomorphic curves in the del Pezzo surface under consideration. The R charge (equivalently the dimension Δ) of the dibaryons that we want to count determines the degree of the associated curve $\mathcal{C}$ by

$$R_{\mathcal{C}} = k\frac{2N}{(9-n)} \tag{8.1}$$

which is analogous to the corresponding expression for bifundamental fields (2.9). As explained in equation (2.11), the degree k is computed as minus the intersection with the canonical class K_n

$$k = -(K_n \cdot \mathcal{C}) = -\left(3H - \sum_i E_i\right) \cdot \mathcal{C} \tag{8.2}$$

The genus of g of $\mathcal{C}$ is related to its degree and self-intersection by the adjunction formula [37]

$$\mathcal{C} \cdot \mathcal{C} = k - 2 + 2g\,. \tag{8.3}$$

Then, in order to have genus greater or equal to zero, we have

$$\mathcal{C} \cdot \mathcal{C} \geq (k-2)\,. \tag{8.4}$$

For computational purposes, it is useful to rewrite this equation in terms of the numbers of D's and E_i's

$$N_D^2 - \sum_i N_{E_i}^2 \geq (k-2)\,. \tag{8.5}$$

Equations (8.1) and (8.5) will be our main tools for the geometric counting. It is important to note that the geometric method can be used not only to provide the multiplicity of states, but also the specific representation of the Weyl group of E_n under which dibaryons transform. This is accomplished by computing their Dynkin components as the intersections of the corresponding curves with the U(1) generators in (2.3). This information can also be contrasted against the results of the algebraic procedure, which is explained in the next section.

8.2 Algebraic counting

The counting of dibaryon operators in the gauge theory can be simplified by exploiting the global symmetry representations assigned to the fields in the quiver. We will refer to this approach as *algebraic counting*. Our discussion applies to the class of toric quivers. Similar ideas for non-toric 3-block quivers will be presented in section 9. The starting point is to look for all possible multiplications of fields that give the appropriate R charge of the dibaryons that we want to consider. In fact, there is no need to take into account all possible combinations, although it is useful to do so in order to check results. The simplicity of this method resides in that it is not necessary to be concerned about the details of how these fields are combined, the proper number of states will be the result of the group structure of the quiver.

Following the discussion in the previous section, we consider one representation for each type of fields and multiply them, yielding for each combination a sum of candidate representations in which dibaryons can in principle transform. The appropriate representation can be distinguished by supplementing the group theory discussion with a simple input consisting of an upper bound on the number of dibaryons coming from inspection of the quiver (some representations are simply too large to correspond to dibaryons formed with fields in the quiver). This representation will in general be generated by some of the alternative combinations considered.

In the following sections, we will perform the geometric counting of dibaryons on the gravity side and compare the results with the ones obtained using the algebraic procedure described above. We will do that for the specific cases of dP_4 and dP_5. These are interesting theories for various reasons. The quiver theories in these cases are fairly non-trivial, with seven and eight gauge groups respectively under which bifundamental chiral multiplets are charged in very different ways. Furthermore, for both geometries, it is possible to perform a cross check between the counting done on different dual gauge theories using the E_n representation structure presented in section 3. Finally, some of these models also contain partial representations, constituing a more subtle check of our techniques.

8.3 Dibaryons in del Pezzo 4

Geometric counting. Let us count the dibaryons of degree $k = 2$ to 5, by using equations (8.2) and (8.5). Degree 1 dibaryons form the fundamental **10** representation of E_4 and had been considered in [34]. The results in this case are

$k=2$		$k=3$		$k=4$		$k=5$	
$(1,\underline{-1,0,0,0})$	4	$(1,0,0,0,0)$	1	$(2,\underline{-1,-1,0,0})$	6	$(2,\underline{-1,0,0,0})$	4
$(2,-1,-1,-1,-1)$	1	$(2,\underline{-1,-1,-1,0})$	4	$(3,\underline{-2,-1,-1,-1})$	4	$(3,\underline{-2,-1,-1,0})$	12
						$(4,\underline{-2,-2,-2,-1})$	4
	$\bar{\mathbf{5}}$		$\mathbf{5}$		$\overline{\mathbf{10}}$		$\mathbf{24}-4$

(8.6)

where we have used the vector notation of (2.2) to indicate divisors, and the convention is that all possible permutations of the underlined entries have to be considered. This notation will be used along the rest of the paper. We list in this table the multiplicities of divisors and the representation in which the curves of each degree, and hence the dibaryons, transform.

The dibaryons at $k = 5$ form a **20** Weyl orbit of E_4. We have indicated it as **24** $-$ 4 in (8.6) to make contact with the more familiar E_4 representations. We see that **20** corresponds to the adjoint **24** of E_4 minus the four Cartan generators, which correspond to dibaryons whose associated divisor would be the canonical class of dP_4.

As explained in section 8.1, we have been able not only to count the multiplicity of states at each k, but also to determine the specific representations, by computing the Dynkin components that correspond to the divisors.

Algebraic counting. The quiver diagrams for the two toric phases of dP_4 were presented in figure 7 and figure 8. Let us now compute the number of dibaryons of each degree using the representation structure derived in section 3.

The R charges of the bifundamental fields are, for Model I,

Representation	R
10	2/5
$\bar{5}$	4/5

(8.7)

and, for Model II,

Representation	R
10	2/5
$\bar{5}_a$	4/5
partial $\bar{5}_b$	4/5
partial 5	6/5

(8.8)

As we discussed in section 2.1, basic representations of E_n coincide with Weyl orbits (with the exception of the adjoint). These are the representations in which dibaryons of small degree are organized. For this reason, whenever we should multiply Weyl orbits in the coming sections, we will be in fact computing products of E_n representations. This is intended to simplfy the computations, since the products of E_n representations can be found in standard references such as [38]. This will generally be sufficient for our purposes in all the cases we will study, since we will only be interested in small representations emerging from such products. We hope the readers will not be mislead by this assumption and will keep it in mind.

The case $k = 2$. According to (8.1), these states have $R = 4N/5$. Let us start by considering Model I. Taking into account how the fields forming the representations above are distributed in the quiver, we see that we can construct these dibaryons in the following ways

$$\left.\begin{array}{l} 10 \otimes 10 = \underline{\bar{5}} \oplus \overline{45} \oplus \overline{50} \\ \underline{\bar{5}} \end{array}\right\} \to \bar{5} \qquad (8.9)$$

It is immediate to discard the $\mathbf{\overline{45}}$ and $\mathbf{\overline{50}}$ representations because of their large dimensions. The resulting one is then the $\mathbf{\bar{5}}$. We have underlined it in (8.9) to indicate how it appears in both alternative ways of constructing these dibaryons.

JHEP07(2004)060

We arrive at the same result when we look at how $R = 4N/5$ dibaryons can be formed in Model II, a dual theory.

$$\left.\begin{array}{l} 10 \otimes 10 = \underline{\bar{5}} \oplus \overline{45} \oplus \overline{50} \\ \underline{\bar{5}_a} \\ \underline{\bar{5}_b} \end{array}\right\} \to \bar{5} \tag{8.10}$$

The case $k = 3$. For $k = 3$, we have $R = 6N/5$. Then the possible ways of constructing these dibaryons are

$$\left.\begin{array}{l} 10 \otimes 10 \otimes 10 = \underline{5} \oplus \bar{5} \oplus 2\ 45 \oplus \overline{45} \oplus 50 \oplus 70 \oplus 157'' \oplus 2\ 280 \\ 10 \otimes \bar{5} = \underline{5} \oplus 45 \end{array}\right\} \to 5 \tag{8.11}$$

By comparing the two possible constructions, one can conclude that the representation will be either a **5**, a **45** or a direct sum of both of them. Quick inspection of the quiver reveals that it is not possible to form at least 45 dibaryons, leaving us with the **5** representation. The same result can be obtained in Model II, where we have

$$\left.\begin{array}{l} 10 \otimes \bar{5}_a = \underline{5} \oplus 45 \\ 10 \otimes \bar{5}_b = \underline{5} \oplus 45 \\ \underline{5} \end{array}\right\} \to 5 \tag{8.12}$$

The case $k = 4$. We have now $R = 8N/5$, and the following possibilities in Model I

$$\left.\begin{array}{l} 10 \otimes 10 \otimes 10 \otimes 10 = \underline{\overline{10}} \oplus \overline{15} \oplus \cdots \\ 10 \otimes 10 \otimes \bar{5} = \underline{\overline{10}} \oplus \overline{15} \oplus \cdots \\ \bar{5} \otimes \bar{5} = \underline{\overline{10}} + \overline{15} \end{array}\right\} \to \overline{10} \tag{8.13}$$

where the dots indicate large representations that are not relevant for our discussion. As in the previous case, both the $\overline{\mathbf{10}}$ and the $\overline{\mathbf{15}}$ seem possible, but the $\overline{15}$ is discarded because of its large multiplicity. Repeating the calculation for Model II, we get

$$\left.\begin{array}{l} 10 \otimes 10 \otimes \bar{5}_a = \underline{\overline{10}} \oplus \overline{15} \oplus \cdots \\ 10 \otimes 10 \otimes \bar{5}_b = \underline{\overline{10}} \oplus \overline{15} \oplus \cdots \\ \bar{5}_a \otimes \bar{5}_a = \underline{\overline{10}} \oplus \overline{15} \\ \bar{5}_b \otimes \bar{5}_a = \underline{\overline{10}} \oplus \overline{15} \\ 10 \otimes 5 = \underline{\overline{10}} \oplus \overline{40} \end{array}\right\} \to \overline{10} \tag{8.14}$$

in coincidence with the computation in Model I.

The case $k = 5$. These dibaryons have $R = 2N$. In Model I, we have the following alternatives to construct them

$$\left.\begin{array}{l} \bar{5} \otimes \bar{5} \otimes 10 = 1 \oplus \underline{24} \oplus \overline{24} \oplus 2\ 75 \oplus 126 \oplus 175' \\ 10 \otimes 10 \otimes 10 \otimes \bar{5} = 1 \oplus 2\ \underline{24} + \overline{24} \oplus \cdots \\ 10 \otimes 10 \otimes 10 \otimes 10 \otimes 10 \end{array}\right\} \to 24\,. \tag{8.15}$$

Applying the same arguments as before, we obtain the **24** representation. We now remember that we multiplied E_n representations for simplicity, but the objects we are actually interested in, are Weyl orbits. Then, we conclude that $k = 5$ dibaryons form the **20** Weyl orbit, which is obtained from the **24** adjoint representation of E_n by removing the Cartan generators. This is again in agreement with the geometric counting. We get the same result in Model II, by considering

$$\left.\begin{array}{l} 10 \otimes 10 \otimes 5 = 1 \oplus 2\ \underline{24} \oplus 75 \oplus 126 \\ 10 \otimes \bar{5}_a \otimes \bar{5}_a = 1 \oplus \underline{24} \oplus \bar{24} \oplus 2\ 75 \oplus 126 \oplus 175' \end{array}\right\} \to 24\,. \tag{8.16}$$

8.4 Dibaryons in del Pezzo 5

There are three toric phases for dP_5, whose quivers are shown in figures 9, 10 and 11. We will perform the algebraic counting for Model I and Model II. Model III differs from Model II simply by the presence of an additional copy of a partial **10** representation, and thus the counting works identically.

Geometric counting. Degree 1 dibaryons were studied in [34] and they form a **16** of E_5. Based on equations (8.2) and (8.5), the geometric procedure determines the following representations for dibaryons of degree 2 to 4

$k=2$		$k=3$		$k=4$	
$(1,\underline{-1,0,0,0,0})$	5	$(1,0,0,0,0,0)$	1	$(2,\underline{-1,-1,0,0,0})$	10
$(2,\underline{-1,-1,-1,-1,0})$	5	$(2,\underline{-1,-1,-1,0,0})$	10	$(3,\underline{-2,-1,-1,-1,0})$	20
		$(3,\underline{-2,-1,-1,-1,-1})$	5	$(4,\underline{-2,-2,-2,-1,-1})$	10
	10		$\bar{\mathbf{16}}$		$\mathbf{45} - 5$

(8.17)

Algebraic counting. The quivers for Models I and II are presented in figure 9 and figure 10. The R charge assignations are

Representation	R
16	$1/2$

(8.18)

for Model I, and

Representation	R
16	$1/2$
partial 10	1

(8.19)

for Model II.

The case $k = 2$. These dibaryons have $R = N$. In Model I, all possible combinations that produce them take the form

$$16 \otimes 16 = \underline{10} \oplus 120 \oplus 126\} \to 10\,. \tag{8.20}$$

The **10** representation has been chosen because it is already clear from the quiver that is not possible to form enough dibaryons to fill the other representations

Moving to Model II, we get two possible ways of forming these dibaryons

$$\left.\begin{array}{l} 16 \otimes 16 = \underline{10} \oplus 120 \oplus 126 \\ \underline{10} \end{array}\right\} \rightarrow 10\,. \tag{8.21}$$

Reproducing the result obtained with Model I. Notice that although we have included all the possible combinations of bifundamental fields giving rise to $k = 2$ dibaryons, the computation actually reduces in this case to noticing that it is possible to construct them by antisymmetrizing fields of a single kind (the ones with $R = 1$) and thus the resulting dibaryons should fill the representation in which these fields transform, i.e. a **10**.

The case $k = 3$. The $k = 3$ dibaryons have $R = 3N/2$. Their construction in Model I corresponds to the combination

$$\left. 16 \otimes 16 \otimes 16 = 2\underline{\overline{16}} \oplus 3\overline{144} \oplus 5\overline{60} \oplus 6\overline{72} \oplus 21\overline{200} \right\} \rightarrow \overline{16}\,. \tag{8.22}$$

Proceeding as in previous sections by considering an upper bound in the number of dibaryons, we conclude that they form a $\overline{\mathbf{16}}$ representation. We can arrive at the same conclusion by studying Model II, where

$$\left.\begin{array}{l} 10 \otimes 16 = \underline{\overline{16}} \oplus \overline{144} \\ 16 \otimes 16 \otimes 16 = 2\ \underline{\overline{16}} \oplus \cdots \end{array}\right\} \rightarrow \overline{16}\,. \tag{8.23}$$

The case $k = 4$. These operators have $R = 2N$ and can be built in Model I in the following way

$$\left. 16 \otimes 16 \otimes 16 \otimes 16 = 1 \oplus 6\underline{45} \oplus \cdots \right\} \rightarrow 45 \tag{8.24}$$

The extra terms in the product correspond extra representations that cannot correspond to dibaryons due to their large dimensions, leaving us with the **45**. Once again, remember that the relevant objects are Weyl orbits, so we conclude that $k = 4$ dibaryons form the **40** Weyl orbit that results when removing the Cartan generators from the adjoint **45**. This result agrees with the geometric counting. Moving to Model II, we get the same result.

$$\left.\begin{array}{l} 10 \otimes 16 \otimes 16 = 1 \oplus 2\underline{45} \oplus \cdots \\ 16 \otimes 16 \otimes 16 \otimes 16 = 1 \oplus 6\ \underline{45} \oplus \cdots \end{array}\right\} \rightarrow 45 \tag{8.25}$$

We have been able to observe a complete match between the geometric and algebraic enumeration of dibaryon operators. The algebraic counting appears as a useful tool for organizing the counting. We will study in the next section a related idea that applies to non-toric quivers.

9. Del Pezzo 7 and 8

The methods we have developed for classifying bifundamental fields in del Pezzo theories based on their relation to the E_n Lie algebras cease to work in a simple way for models in which the gauge group factors have different ranks. The main reason was pointed out in section 2, where we noted that equations (2.8) and (2.9) predict in these theories fractional U(1) charges that cannot be identified with Dynkin coefficients of E_n representations.

Dibaryon operators correspond to the antisymmetrized product of a large number of bifundamental fields and do transform into E_n representations. Moreover, we discussed in section 7 how, for the special case of three block models, a maximal $\mathrm{SU}(\alpha)\times\mathrm{SU}(\beta)\times\mathrm{SU}(\gamma)$ subgroup of E_n becomes manifest. This subgroup will play a similar role to the one of E_n in the algebraic counting of dibaryons in 3-block non-toric quivers. Bifundamental fields can be organized into representations of the maximal subgroup which can be used to determine the representations of dibaryons.

Contrary to what happens in toric quivers, the representations of the maximal subgroup in which dibaryons transform are not determined by straigthforward multiplication of represenations of constituent bifundamental fields, but result from the application of the following procedure. For each of the factors $\mathrm{SU}(\alpha)\times\mathrm{SU}(\beta)\times\mathrm{SU}(\gamma)$, the appropriate representation is determined by the nodes that are contracted with antisymmetric tensors (i.e. nodes that are at heads or tails of the quiver path defining the dibaryon, and that are not contracted with fundamental or antifundamental indices of other bifundamental fields). We have to multiply one $\mathrm{SU}(\alpha)$ representation for each of these 'free nodes', and take the antisymmetric combination in the product. This is the main difference with traditional multiplication of representations, since the number of 'free nodes' for each $\mathrm{SU}(\alpha)$ can be different.

9.1 Del Pezzo 7

In the case of dP_7, there are three minimal solutions to the Diophantine equation (7.2), from which all others can be generated through Seiberg dualities: $(\alpha,\beta,\gamma)=(1,1,8)$, $(2,4,4)$ and $(1,3,6)$ [39]. We will examine now the $(2,4,4)$ model. Figure 28 shows the quiver diagram for this theory.

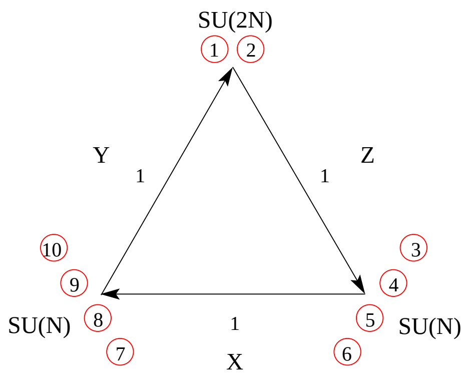

Figure 28: One of the possible 3-block quivers for dP_7.

The gauge groups for nodes in each of the three blocks are $\mathrm{U}(2N)$, $\mathrm{U}(N)$ and $\mathrm{U}(N)$. The maximal subgroup of E_7 is in this case $\mathrm{SU}(2)\times\mathrm{SU}(4)\times\mathrm{SU}(4)$, under which bifundamental fields have the following transformation properties

$$\begin{aligned} X &: (1,4,\bar{4}) \\ Y &: (2,1,4) \\ Z &: (2,\bar{4},1) \end{aligned} \qquad (9.1)$$

Their R-charges are

$$R_X = 1 \qquad R_Y = R_Z = \frac{1}{2}\,. \qquad (9.2)$$

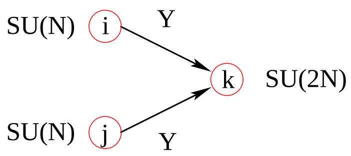

Figure 29: Level one Y^2 dibaryons in the $(2,4,4)$ dP_7 3-block model.

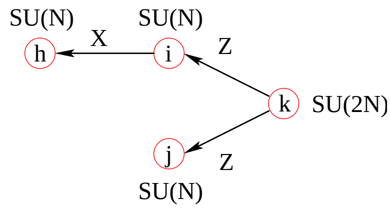

Figure 30: Level two Z^2X dibaryons in the $(2,4,4)$ dP_7 3-block model.

Before moving on to count dibaryons in this theory, let us consider in detail two explicit examples, in order to gain familiarity with the application of the rules described above. The first example corresponds to the level one dibaryons constructed by antisymmetrizing $2N$ Y fields. Suppressing the N factor in the multiplicity, which will be common to all dibaryons, we will refer to them as Y^2 dibaryons.

In the figure, $k \in \{1,2\}$ and $i,j \in \{7,8,9,10\}$. The Y fields transform trivially under the first SU(4). We have only one node in the SU(2) block, and thus the Y^2 dibaryons will transform under it in the **2** representation. Nodes i and j transform in the **4** of the second SU(4) and thus we need to take the antisymmetric combination in the product of two **4**'s, giving the **6**. In summary, Y^2 dibaryons form the $(2,1,6)$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$.

Let us now consider the level two dibaryons constructed from $2N$ Z and N X fields. As before, we refer to them as Z^2X. They correspond to paths in the quiver of the form

Here $h \in \{7,8,9,10\}$, $i,j \in \{3,4,5,6\}$ and $k \in \{1,2\}$. There is only one node for each $\mathrm{SU}(\alpha)$ that is contracted with epsilon tensors. For SU(2) this corresponds to head of Z fields, the first SU(4) corresponds to the tail of Z and the second SU(4), to the tail of X. Using that X transforms as $(1,4,\bar{4})$ and Z as $(2,\bar{4},1)$ we immediately conclude that the Z^2X dibaryons form the $(2,\bar{4},4)$ representation.

Having illustrated the practical details of the classification into maximal subgroups with two examples, we can move on and write down the combinations of fields in the gauge theory that give rise to dibaryons of different levels explicitly. Level one dibaryons have R-charge N and fill the fundamental **56** of E_7 [34]. They can be organized as shown in the table below.

Dibaryon	Number of states	$\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4)$
X	16	$(1,4,\bar{4})$
YZ	16	$(1,\bar{4},4)$
Y^2	12	$(2,1,6)$
Z^2	12	$(2,6,1)$

(9.3)

Our results correspond to the branching of the **56** representation under the maximal subalgebra

$$56 \to (1,4,\bar{4}) + (1,\bar{4},4) + (2,1,6) + (2,6,1)\,. \tag{9.4}$$

At level two, the dibaryons have R-charge $2N$. The geometric counting gives

Divisor	Number of states
$(1,\underline{-1,0,0,0,0,0,0})$	7
$(2,\underline{-1,-1,-1,-1,0,0,0})$	35
$(3,\underline{-2,-1,-1,-1,-1,-1,0})$	42
$(4,\underline{-2,-2,-2,-1,-1,-1,-1})$	35
$(5,\underline{-2,-2,-2,-2,-2,-2,-1})$	7
Total	**133** – 7

(9.5)

corresponding to part of the adjoint representation of E_7, which has dimension 133. The remaining seven elements correspond to the Cartan generators of E_7. The counting on the gauge theory is shown in the next table.

Dibaryon	Number of states	$\mathrm{SU}(2)\times\mathrm{SU}(4)\times\mathrm{SU}(4)$
ZXY	2	$(3,1,1)-1$
XYZ	12	$(1,15,1)-3$
YXZ	12	$(1,1,15)-3$
XY^2	32	$(2,4,4)$
$\mathrm{Z}^2\mathrm{X}$	32	$(2,\bar{4},\bar{4})$
$\mathrm{Y}^2\mathrm{Z}^2$	36	$(1,6,6)$
Total	126	**133** – 7

(9.6)

where we have removed the Cartan generators from each of the adjoint representations. The result is in complete agreement with the geometric counting. We see that dibaryons are classified under the $\mathrm{SU}(2)\times\mathrm{SU}(4)\times\mathrm{SU}(4)$ maximal subgroup according to the branching of the **133** E_7 representation

$$133 \to (3,1,1)+(1,15,1)+(1,1,15)+(2,4,4)+(2,\bar{4},\bar{4})+(1,6,6)\,. \qquad (9.7)$$

9.2 Del Pezzo 8

The eigth del Pezzo can be studied using the same methodology. The 3-block Diophantine equation for dP_8 has four minimal solutions: $(\alpha,\beta,\gamma)=(1,1,9)$, $(1,2,8)$, $(2,3,6)$ and $(1,5,5)$. The $(1,2,8)$ case has been studied in [19], where the 240 level one dibaryons were written explicitly in the gauge theory. Let us apply maximal subgroup decomposition to classify level one dibaryons in the $(1,1,9)$ model. The quiver for this theory is shown in figure 31.

The maximal subgroup of E_8 is in this case SU(9). All bifundamental fields in this model have R-charge 2/3.[8] Proceeding as before, we present in next table the 240 level

[8]As a curiosity note that this quiver is precisely that of the $C^3/\Delta(27)$ orbifold [40, 41].

JHEP07(2004)060

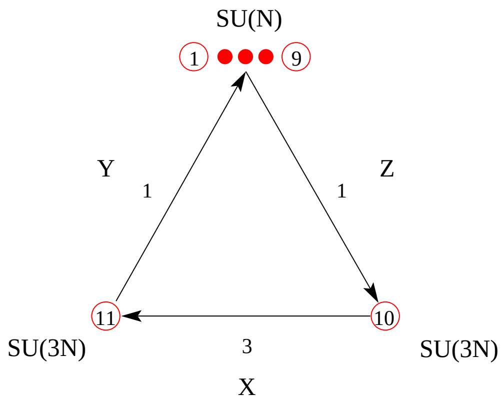

Figure 31: One of the possible 3-block quivers for dP_8.

one dibaryons, along with their tranformation properties under SU(9).

$$\begin{array}{|c|c|c|}\hline \text{Dibaryon} & \text{Number of states} & \mathrm{SU}(9) \\ \hline \mathrm{ZXY} & 72 & 80-8 \\ \mathrm{Y}^3 & 84 & \overline{84} \\ \mathrm{Z}^3 & 84 & 84 \\ \hline \textbf{Total} & 240 & \mathbf{248}-8 \\ \hline \end{array} \tag{9.8}$$

10. Conclusions

In this paper we have identified the global symmetries of the gauge theories on D3-branes probing complex cones over del Pezzo surfaces, which have their origin in the automorphism of the underlying geometry. This has been possible due to the association of divisors in the del Pezzo surface to every bifundamental field in the quiver. The correspondence between bifundamental fields and divisors follows from studies of a special class of dibaryon operators. Each of them is constructed by antisymmetrizing various copies of a single field in the quiver. For each dP_n, the bifundamental matter of the theories has been explicitly organized in irreducible representations of E_n. We presented the results in section 3. This classification has been obscure in the past due to the fact that, in general, irreducible representation of the global symmetry group are formed by bifundamental fields charged under different pairs of gauge groups.

We encountered some theories in which the matter content seems, at a first glance, insufficient to complete representations. We discussed how all the models can be naturally studied within the same framework. The fields that appear to be absent from partial representations sit in bidirectional arrows in the quiver (i.e. quadratic gauge invariants) and that also form quadratic invariants under the global symmetry group. Thus, following the same rules that apply to all other cases, mass terms for these fields are present in the superpotential, and they are integrated out when considering the low energy physics. The geometric origin of partial representations was discussed in section 4.1, where we explained how to determine the location in the quiver of the massive fields.

The E_n classification of the models becomes particularly helpful in writing down superpotentials, both for toric and non-toric del Pezzos. The basic elements of their construction are the gauge invariant projections of the singlets under the action of the global symmetry group. We have seen how superpotentials become completly determined by this principle (and, in a few cases, information about the higgsing from a higher dP_n).

The blow-down of a 2-cycle takes us from dP_n to dP_{n-1}. This geometric action translates on the gauge theory side to a non-zero VEV for a bifundamental field that higgses the quiver. We have shown in section 5 how to use the group theory classification of the quiver to identify the bifundamental fields that do the correct job. By turning on a VEV for any field in the fundamental representation of E_n, it is higgsed down to one of E_{n-1} and a dP_{n-1} quiver is produced. In this way, we have presented a clear systematic prescription that identifies all possible ways in which a dP_n quiver can be higgsed to obtain another quiver that corresponds to dP_{n-1}.

It would be interesting to extend the discussion of this paper to gauge theories on D3-branes probing different singularities, in which other groups of automorphisms will point towards global symmetries of the corresponding field theories. In the case that these symmetry groups include or are included in the ones for del Pezzo theories, the group theory concepts of section 5.5 would indicate how to derive those theories by (un)higgsing.

In section 6, we presented a simple set of rules that determine how the representation structure of a gauge theory transform under Seiberg duality. This was used to rederive the classification based on divisors obtained in section 3.

We showed in section 8 how the gauge theory counting of dibaryon operators is organized and simplified by using the E_n structure. We further verified that the algebraic counting in the field theory match the geometric counting of curves of different degrees, in agreement with the AdS/CFT correspondence.

Finally, in section 9, we used maximal subgroups of E_n to classify and count dibaryons in non-toric 3-block quivers for dP_7 and dP_8.

Acknowledgments

The authors would like to thank Y.H. He, C. Herzog and J. Walcher for various insightful discussions and comments on early versions of the manuscript. S.F. also thanks M. Schvellinger for helplful conversations. A.H. appreciates the hospitality of the Institute for Advanced Study during the preparation of this work. This Research was supported in part by the CTP and LNS of MIT and the U.S. Department of Energy under cooperative research agreement # DE-FC02-94ER40818, and by BSF – American-Israeli Bi-National Science Foundation. A.H. is also indebted to the Reed Fund Award and a DOE OJI ward.

References

[1] C. Beasley, B.R. Greene, C.I. Lazaroiu and M.R. Plesser, *D3-branes on partial resolutions of abelian quotient singularities of Calabi-Yau threefolds*, *Nucl. Phys.* **B 566** (2000) 599 [hep-th/9907186].

JHEP07(2004)060

[2] B. Feng, A. Hanany and Y.-H. He, *D-brane gauge theories from toric singularities and toric duality*, *Nucl. Phys.* **B 595** (2001) 165 [hep-th/0003085].

[3] B. Feng, A. Hanany and Y.-H. He, *Phase structure of D-brane gauge theories and toric duality*, *J. High Energy Phys.* **08** (2001) 040 [hep-th/0104259].

[4] B. Feng, A. Hanany, Y.-H. He and A.M. Uranga, *Toric duality as Seiberg duality and brane diamonds*, *J. High Energy Phys.* **12** (2001) 035 [hep-th/0109063].

[5] C.E. Beasley and M.R. Plesser, *Toric duality is Seiberg duality*, *J. High Energy Phys.* **12** (2001) 001 [hep-th/0109053].

[6] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[7] K. Intriligator and B. Wecht, *Baryon charges in 4d superconformal field theories and their AdS duals*, *Commun. Math. Phys.* **245** (2004) 407 [hep-th/0305046].

[8] K.A. Intriligator and N. Seiberg, *Mirror symmetry in three dimensional gauge theories*, *Phys. Lett.* **B 387** (1996) 513 [hep-th/9607207].

[9] M. Porrati and A. Zaffaroni, *M-theory origin of mirror symmetry in three dimensional gauge theories*, *Nucl. Phys.* **B 490** (1997) 107 [hep-th/9611201].

[10] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[11] N. Seiberg, *Five dimensional SUSY field theories, non-trivial fixed points and string dynamics*, *Phys. Lett.* **B 388** (1996) 753 [hep-th/9608111].

[12] K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys.* **B 497** (1997) 56 [hep-th/9702198].

[13] B. Feng and A. Hanany, *Mirror symmetry by O3-planes*, *J. High Energy Phys.* **11** (2000) 033 [hep-th/0004092].

[14] O. Aharony and A. Hanany, *Branes, superpotentials and superconformal fixed points*, *Nucl. Phys.* **B 504** (1997) 239 [hep-th/9704170].

[15] O. Aharony, A. Hanany and B. Kol, *Webs of* (p,q)*5-branes, five dimensional field theories and grid diagrams*, *J. High Energy Phys.* **01** (1998) 002 [hep-th/9710116].

[16] O. DeWolfe, A. Hanany, A. Iqbal and E. Katz, *Five-branes, seven-branes and five-dimensional* E_n *field theories*, *J. High Energy Phys.* **03** (1999) 006 [hep-th/9902179].

[17] M. Wijnholt, *Large volume perspective on branes at singularities*, *Adv. Theor. Math. Phys.* **7** (2004) 1117 [hep-th/0212021].

[18] A. Iqbal, A. Neitzke and C. Vafa, *A mysterious duality*, *Adv. Theor. Math. Phys.* **5** (2002) 769 [hep-th/0111068].

[19] C.P. Herzog and J. Walcher, *Dibaryons from exceptional collections*, *J. High Energy Phys.* **09** (2003) 060 [hep-th/0306298].

[20] A. Hanany and J. Walcher, *On duality walls in string theory*, *J. High Energy Phys.* **06** (2003) 055 [hep-th/0301231].

[21] S. Franco, A. Hanany, Y.-H. He and P. Kazakopoulos, *Duality walls, duality trees and fractional branes*, hep-th/0306092.

[22] S. Franco, A. Hanany and Y.-H. He, *A trio of dualities: walls, trees and cascades*, *Fortschr. Phys.* **52** (2004) 540 [hep-th/0312222].

[23] S. Franco, Y.-H. He, C. Herzog and J. Walcher, *Chaotic duality in string theory*, hep-th/0402120.

[24] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[25] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[26] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[27] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an $N = 1$ superconformal gauge theory*, *Phys. Rev.* **D 58** (1998) 125025 [hep-th/9808075].

[28] S. Gukov, M. Rangamani and E. Witten, *Dibaryons, strings and branes in AdS orbifold models*, *J. High Energy Phys.* **12** (1998) 025 [hep-th/9811048].

[29] B. Feng, S. Franco, A. Hanany and Y.-H. He, *Symmetries of toric duality*, *J. High Energy Phys.* **12** (2002) 076 [hep-th/0205144].

[30] B. Feng, S. Franco, A. Hanany and Y.-H. He, *Unhiggsing the del Pezzo*, *J. High Energy Phys.* **08** (2003) 058 [hep-th/0209228].

[31] A. Hanany and A. Iqbal, *Quiver theories from D6-branes via mirror symmetry*, *J. High Energy Phys.* **04** (2002) 009 [hep-th/0108137].

[32] S. Franco and A. Hanany, *Geometric dualities in 4d field theories and their 5d interpretation*, *J. High Energy Phys.* **04** (2003) 043 [hep-th/0207006].

[33] B. Feng, Y.-H. He and F. Lam, *On correspondences between toric singularities and (p, q)-webs*, hep-th/0403133.

[34] C.P. Herzog and J. McKernan, *Dibaryon spectroscopy*, *J. High Energy Phys.* **08** (2003) 054 [hep-th/0305048].

[35] C.P. Herzog, *Exceptional collections and del Pezzo gauge theories*, *J. High Energy Phys.* **04** (2004) 069 [hep-th/0310262].

[36] B. Feng, A. Hanany, Y.H. He and A. Iqbal, *Quiver theories, soliton spectra and Picard-Lefschetz transformations*, *J. High Energy Phys.* **02** (2003) 056 [hep-th/0206152].

[37] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons Inc., New York 1994, reprint of the 1978 original.

[38] R. Slansky, *Group theory for unified model building*, *Phys. Rept.* **79** (1981) 1.

[39] B.V. Karpov and D.Yu. Nogin, *Three-block exceptional collections over del Pezzo surfaces*, *Izv. Ross. Nauk Ser. Mat.* **62** (1998) no. 3, 3, translation in *Izv. Ross. Math.* **62** (1998) no. 3, 429 [alg-geom/9703027].

[40] A. Hanany and Y.-H. He, *Non-abelian finite gauge theories*, *J. High Energy Phys.* **02** (1999) 013 [hep-th/9811183].

[41] B. Feng, A. Hanany, Y.-H. He and N. Prezas, *Discrete torsion, covering groups and quiver diagrams*, *J. High Energy Phys.* **04** (2001) 037 [hep-th/0011192].

JHEP07(2004)060